THE NUMERICAL RANGE OF AN OPERATOR

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Candidate's declaration

I hereby certify that this thesis has been composed by myself and comprises my own work.

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Abstract.

Numerical range for operators on Hilbert space has been studied for some time. The concept has been extended to operators on a general normed space by G. Lumer and F.L. Bauer, and to elements of a normed algebra by F.F. Bonsall. This forms the subject of the thesis.

The relationship between the numerical range and the spectrum is studied. It is shown that some of the properties of operators on Hilbert space in this connection remain true for Banach space.

Estimates for the norm of an operator and its powers in terms of the size of its numerical range are given, both for real and complex spaces. A consequence of the ratio of norm to numerical radius for an operator on a complex space being a maximum is obtained.

Operators on a complex space which have real numerical range (Hermitian operators) are considered. Although an example shows that the square of a Hermitian operator need not be Hermitian, some properties of such squares are found.

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Table of contents.

Chapter 1

INTRODUCTION

Toeplitz [27], in 1918, was the first to consider a numerical range of an operator, in this case an $n \times n$ matrix considered as acting on n-dimensional complex Euclidean space. Given that the matrix is A, he defined its numerical range $W(A)$ to be $\{x^*Ax : x \in C^n, x^*x = 1\}$, where x denotes a column vector, and x^* is its complex conjugate transpose. He proved that $W(A)$ contains the eigenvalues of A, and that the boundary of the unbounded component of the complement of $W(A)$ is a convex curve. He showed that $W(A)$ is equal to the convex hull of the eigenvalues of A when A is normal, but did not decide whether $W(A)$ is convex in general. For $x \in C^n$, define the norm of x as $\|x\| = (x^*x)^{\frac{1}{2}}$, and $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$. Put $w(A) = \sup\{|z| : z \in W(A)\}$. Then he proved that $\|A\| \leq 2w(A)$. In 1919, Hausdorff [15] proved that $W(A)$ is, in fact, convex for any A.

The definition of numerical range was extended in the obvious way to operators on an infinite dimensional Hilbert space.

DEFINITION 1.1. Let T be a bounded linear operator on a Hilbert space X. Then the numerical range $W(T)$ of T is defined as

$$W(T) = \{(Tx,x) : x \in X, \|x\| = 1\}.$$

This definition coincides with that of Toeplitz in the finite dimensional case. The same proofs as before show that $W(T)$ is convex, and that $\|T\| \leq 2w(T)$. It was also proved that $sp(T) \subset W(T)^{-}$, where $sp(T)$ denotes the spectrum of T, and bar denotes set closure. It also remains true that $W(T)^{-}$ is equal to the convex hull of $sp(T)$, given that T is a normal operator.

In order to extend the definition of numerical range to an operator on a general normed space, Lumer [17] defined a semi-inner-product on a normed space. First, he defines a semi-inner-product space, as follows.

DEFINITION 2. A semi-inner-product space is a vector space over a field F, which is either R or C, together with a mapping $[\ ,\ ] : X \times X \to F$ which satisfies, for $x, y, z \in X$ and $a \in F$,

$$[x+y, z] = [x, z] + [y, z]$$
$$[ax, y] = a[x, y]$$
$$[x, x] > 0 \qquad (x \neq 0)$$
$$|[x, y]|^2 \leq [x, x]\,[y, y]$$

He proves that every semi-inner-product space is a normed linear space with the norm $[x, x]^{\frac{1}{2}}$. He also shows that, given any normed linear space X, a semi-inner-product may be defined on X in such a way that the norm induced on X by the semi-inner-product is equal to the given norm. To see this, note that for each $x \in X$, there exists $f_x \in X'$ such that $\|f_x\| = \|x\|$ and $f_x(x) = \|x\|^2$. By the axiom of choice, assume such an f_x found for each $x \in X$. Then, defining $[x, y] =$

$f_y(x)$ $(x, y \in X)$, the conditions of definition 1 are satisfied, and $[x,x]^{\frac{1}{2}} = \|x\|$ $(x \in X)$. Such a semi-inner-product on a normed space is said to be consistent with the norm.

We may now define the numerical range of an operator, by analogy with the Hilbert space definition.

DEFINITION 3. Given a semi-inner-product space X, and T a bounded linear operator on X, the numerical range $W(T)$ of T is defined by

$$W(T) = \left\{ [Tx, x] : x \in X, \ [x,x] = 1 \right\}.$$

Given that X is a Hilbert space, the only semi-inner-product on X consistent with the norm is the inner product. The numerical range $W(T)$ of an operator T on a Hilbert space X defined in terms of this semi-inner-product is therefore the same as the numerical range as given in definition 1.

In general, the numerical range $W(T)$ is not unique, as there may be many semi-inner-products on X consistent with the norm. $W(T)$ need not be convex, or even connected. It is not known whether it is still true that $sp(T) \subset W(T)^-$ for an operator T on a complex space. It is easy to see that eigenvalues of T are contained in $W(T)$. Lumer has shown that also the approximate point spectrum of T is contained in $W(T)^-$. For a normed space X, $B(X)$ denotes the space of all bounded linear operators on X. S denotes the unit sphere $\{x \in X : \|x\| = 1\}$.

DEFINITION 4. Let X be a normed space, and $T \in B(X)$. The approximate point spectrum $\pi(T)$ is defined as

$$\pi(T) = \left\{ z \in F : \inf_{x \in S} \|(T-z)x\| = 0 \right\}.$$

It is clear that $\pi(T) \subset sp(T)$. We shall require the following result, which follows immediately from Rickart [23, theorem 1.5.4.].

THEOREM 5. Let X be a complex Banach space, and $T \in B(X)$. Then $\partial sp(T) \subset \pi(T)$.

We shall show later that, when X and X' are each equipped with semi-inner-products consistent with the norm, and X is complete, then

$$sp(T) \subset W(T)^{-} \cup W(T^{*}).$$

A different approach to the generalisation of numerical range to operators on any finite dimensional normed space was adopted by F.L. Bauer [1]. The numerical range resulting became known as "the Bauer field of values of a matrix". Bonsall, Cain and Schneider [6] showed how this definition could be extended to an operator on a general normed linear space. The construction in both cases may be described as follows.

Let X be a normed linear space over a field F, which is either R or C. Let X' denote the dual space of all continuous linear functionals on X. Let $S = \{ x \in X : \|x\| = 1 \}$ and $S' = \{ f \in X' : \|f\| = 1 \}$. For $x \in S$ define the set $D(x)$ of support functionals of x by

$$D(x) = \{f \in S' : f(x) = 1\}$$

$D(x)$ is non-empty, by the Hahn-Banach theorem. Suppose that T is a bounded linear operator on X. For $x \in S$, define $V(T,x)$ by

$$V(T,x) = \{f(Tx) : f \in D(x)\}.$$

We now define the numerical range $V(T)$ of T as

$$V(T) = \cup \{V(T,x) : x \in S\}.$$

The definition may be expressed in a slightly different form. Let P be defined by

$$P = \{(x,f) : x \in S, f \in D(x)\},$$

so that P is a subset of $X \times X'$. Then

$$V(T) = \{f(Tx) : (x,f) \in P\}.$$

It is convenient to write $v(T)$ for $\sup \{|z| : z \in V(T)\}$, which is called the numerical radius of T.

For any normed space X, and bounded linear operator T on X, a numerical range $W(T)$ of T results from choosing, for each $x \in S$, a single element $f_x \in D(x)$, and setting $W(T) = \{f_x(Tx) : x \in S\}$. $W(T)$ is not unique, but depends on the choice of semi-inner-product on X. $V(T)$ differs in that, for each $x \in S$, every functional in $D(x)$ is used. $V(T)$ is therefore equal to the union of all possible numerical ranges $W(T)$ of T.

Given that X is a Hilbert space, we have $D(x) = \{x\}$ for $x \in S$, and $V(T)$ coincides with the classical numerical range $W(T)$.

2. Basic results.

It is easy to see that $V(T)$ contains the eigenvalues of T. Williams [29] has proved the following theorem.

THEOREM 1.6. Let X be a Banach space over C, and $T \in B(X)$. Then

$$sp(T) \subset V(T)^{-}.$$

$V(T)$ need not be convex, as Nirschl and Schneider [20] showed. However, Bonsall, Cain and Schneider [6] have proved the following theorem.

THEOREM 1.7. Let X be a normed space over F, and $T \in B(X)$. Then $V(T)$ is connected.

It is not known whether $V(T)$ is always simply connected.

Further information about the structure of $V(T)$ has recently been provided by Zenger [31]. He showed that, in the case of an operator T on a finite dimensional complex space, $V(T)$ contains the convex hull of the eigenvalues of T. In theorem 2.3, we show from this that, given a bounded linear operator T on an infinite dimensional complex normed space, $V(T)^{-}$ contains the convex hull of the spectrum of T.

The following elementary results, due to Lumer, are often used.

THEOREM 1.8. Let X be a normed space, and let $T,R \in B(X)$, $a,b \in F$. Then

$$V(aT+b) = aV(T)+b, \quad V(T+R) \subset V(T)+V(R),$$

$$v(aT) = |a|\, v(T), \quad v(T+R) \leq v(T)+v(R).$$

The following theorem, which is proved by Lumer [17] and Bohnenblust and Karlin [4], will be used often. It is stated for a normed space equipped with a semi-inner-product consistent with the norm, but is valid for any normed space when the formulae involving $W(T)$ or $w(T)$ are omitted. The formulae involving exponentials only apply when the space is complete.

THEOREM 1.9. Let X be a normed space equipped with a semi-inner-product consistent with the norm. Let $T \in B(X)$. Then

1. $$\sup\{\operatorname{Re} z : z \in W(T)\} = \lim_{a\to 0+} \frac{\|I+aT\|-1}{a} = \inf_{a>0} \frac{\|I+aT\|-1}{a}$$
$$= \lim_{a\to 0+} \frac{\log\|\exp(aT)\|}{a} = \sup_{a>0} \frac{\log\|\exp(aT)\|}{a} = \sup\{\operatorname{Re} z : z \in V(T)\}.$$

2. $$w(T) = v(T) = \sup_{a\in C\setminus\{0\}} \frac{\log\|\exp(aT)\|}{|a|}$$

3. $$\sup\{|\operatorname{Re} z| : z \in W(T)\} = \sup\{|\operatorname{Re} z| : z \in V(T)\} =$$
$$\sup_{a\in R\setminus\{0\}} \frac{\log\|\exp(aT)\|}{|a|}$$

4. $$\|\exp(aT)\| \le \exp(|a|\,v(T)) \qquad (a \in C).$$

Proof. Parts 1 and 2 are proved in Lumer [17] and Bohnenblust and Karlin [4]. Part 4 follows immediately from part 2. For part 3,

$$\sup\{-\operatorname{Re} z : z \in W(T)\} = \sup\{\operatorname{Re} z : z \in W(-T)\} =$$
$$\sup_{a>0} \frac{\log\|\exp(-aT)\|}{a} = \sup_{b<0} \frac{\log\|\exp(bT)\|}{|b|}$$

Hence $\sup\{|\mathrm{Re}\, z| : z \in W(T)\} = \max(\sup\{\mathrm{Re}\, z : z \in W(T)\}$, $\sup\{-\mathrm{Re}\, z : z \in W(T)\}) = \sup\limits_{a \in R\setminus(0)} \dfrac{\log \|\exp(aT)\|}{|a|}$

It is clear that the same argument goes through for V replacing W.

Let X be a normed space with a semi-inner-product consistent with the norm. From theorem 9, $\sup\{\mathrm{Re}\, z : z \in W(aT)\} = \sup\{\mathrm{Re}\, z : z \in V(aT)\}$ for any $a \in C$ with $|a| = 1$. This implies that $\overline{co}\, W(T) = \overline{co}\, V(T)$, where $\overline{co}\, A$ denotes the closed convex hull of A. Evidently also, given that X has two semi-inner-products consistent with the norm, giving T the numerical ranges $W_1(T)$ and $W_2(T)$, we have $\overline{co}\, W_1(T) = \overline{co}\, W_2(T)$. A further important consequence of theorem 9 is as follows. Suppose that X is a normed linear space, and that $T \in B(X)$. Let $\widetilde{X}$ be the completion of X, and let $\widetilde{T}$ be the extension of T to $\widetilde{X}$. Then $\sup\{\mathrm{Re}\, z : z \in V(\widetilde{T})\} = \inf\limits_{a>0} \dfrac{\|I+a\widetilde{T}\| - 1}{a} = \inf\limits_{a>0} \dfrac{\|I+aT\| - 1}{a}$ $= \sup\{\mathrm{Re}\, z : z \in V(T)\}$. Similarly, $v(\widetilde{T}) = v(T)$. Thus, on passing to the completion of a space, both the norm and the numerical radius of an operator are unchanged. This fact simplifies some proofs.

The following two theorems were proved by Bonsall, Cain and Schneider.

THEOREM 1.10. The mapping $x \to V(T,x)$ is upper-semi-continuous from the unit sphere S with the norm topology into subsets of F.

THEOREM 1.11. Let $X \neq R$, and let $X \times X'$ be topologised by the product of the norm topology and the weak* topology. Then P is connected as a subset of $X \times X'$.

We shall show later that P is connected in the norm x norm topology of $X \times X'$, given that X is a subspace, satisfying a certain condition, of the functions on a set.

Nirschl and Schneider [20] proved that, for an operator T on a finite dimensional space, $\overline{co}\, V(T) = \overline{co}\, sp(T)$ implies that the eigenvalues of T on the boundary of the convex hull of $V(T)$ have ascent (index) one. In fact, their proof shows that, for any operator T, an eigenvalue on the boundary of the convex hull of $V(T)$ has ascent one. In theorem 2.5 we show that the same conclusion holds for an eigenvalue on the boundary of $V(T)$ itself, and give an estimate for the radius of a disc centred on an eigenvalue of ascent greater than one which must be contained in $V(T)$.

For a bounded operator T on a complex Hilbert space, Berger [2] has proved that $w(T^n) \leq w(T)^n$. This is known as the power inequality for the numerical radius. An elementary proof was given by Pearcy [22]. It is not known whether this remains true for a general normed space. We shall show that

$$\|T^n\| \leq en^{\frac{1}{2}}v(T)^n$$

given $T \in B(X)$, X a complex normed space. This is a very much better estimate for $\|T^n\|$ than $\|T^n\| \leq e^n v(T)^n$, which comes from the case $n = 1$.

It remains an open question whether

$$\|T^n\| = O((v(T))^n),$$

and we shall show that this is the case for all meromorphic operators, and in particular therefore for all operators on finite dimensional complex normed spaces.

Bohnenblust and Karlin showed that, for complex space, $\|T\| \leq ev(T)$. However, for real normed space, we can have $v(T) = 0$ and $T \neq 0$. Bonsall and Duncan [7] proved that, for an operator T on real space, $v(T) = v(T^2) = 0$ implies $T = 0$. It is natural to ask whether an inequality of the form $\|T\| \leq av(T) + bv(T^2)^{\frac{1}{2}}$ might hold for real spaces, where a and b are real constants. Lumer has shown that such an inequality does hold. In theorem 3.12 we adapt his argument to obtain definite constants in the inequality. In fact, we show that

$$\|T\| \leq \max\,(48v(T),\ 24v(T^2)^{\frac{1}{2}}).$$

It will be shown that $v(T) = v(T^4) = 0$, for an operator T on real space, implies $T = 0$. It is not known whether an inequality analagous to the above exists in this case also.

In chapter 4, we study Hermitian operators which are the operators on a complex space which have real numerical range.

3. Numerical range for a normed algebra.

Lumer also applied semi-inner-products and numerical range to a normed algebra, as follows. Let

A be a normed algebra over F. Regarding A as a normed linear space, we may define a semi-inner-product on A consistent with the norm. Then the numerical range $W(a)$ of an element $a \in A$ is defined as $\{[ab,b] : b \in A, \|b\| = 1\}$. $W(a)$ is therefore the numerical range of the bounded linear operator T_a on A defined by $T_a b = ab \quad (b \in A)$.

In the same way, Bonsall [5] has extended the Bauer field of values to define a numerical range for an element of a normed algebra A as follows. For $a \in A$, define the numerical range $V(a)$ of a as $V(a) = \{f(ab) : b \in A,\ f \in A',\ \|b\| = \|f\| = f(b) = 1\}$. Again, $V(a)$ is the numerical range $V(T_a)$ of the operator T_a.

Given that A has an identity of norm one, the situation becomes simpler. Then, for $a \in A$, $V(a) = \{f(a) : f \in D(1)\}$. In this case $V(a)$ is always compact and convex.

For an operator T on a normed space X, we can consider T as an element of the normed algebra $B(X)$, so giving rise to a numerical range $V'(T)$ say. The question arises: what is the relationship between $V'(T)$ and $V(T)$, the operator numerical range. In fact, as Bonsall has shown, $V'(T)$ is the closed convex hull of $V(T)$. For, by the remark earlier, $V'(T)$ is closed and convex, and since, for $c \in C$ and $|c| = 1$, $\sup\{\mathrm{Re}\, z : z \in V(cT)\} = \inf_{a>0} \dfrac{\|I+acT\| - 1}{a} =$ $\sup\{\mathrm{Re}\, z : z \in V'(cT)\}$ from theorem 9. We clearly also have $v'(T) = v(T)$.

It is sometimes convenient to prove a result for a normed algebra, and then reinterpret it for an operator T on a normed space X by applying the result to the algebra $B(X)$.

Chapter 2.

Numerical range and the spectrum.

§1.

In this chapter, we shall consider the relationship between the numerical range and the spectrum of an operator. As we have seen, given a bounded linear operator T on a complex Hilbert space, we have $sp(T) \subset W(T)^-$. Bonsall (unpublished) showed that, for an operator T on a complex Banach space, $sp(T) \subset V(T^*)^-$. We now obtain a similar result concerning the Lumer numerical range. The first part is due to Lumer.

THEOREM 2.1. Let X be a complex Banach space with $T \in B(X)$, and let X and X' be equipped with any semi-inner-products consistent with the norm. Then $\pi(T) \subset W(T)^-$ and $sp(T) \setminus \pi(T) \subset W(T^*)$.

<u>Proof</u>. Suppose that $z \in \pi(T)$. Given any $\epsilon > 0$, there exists $x \in S$ such that $\|(T-z)x\| < \epsilon$. Therefore $|[Tx,x] - z| = |[(T-z)x,x]| < \epsilon$. Hence $z \in W(T)^-$. Now assume that $z \in sp(T) \setminus \pi(T)$. Then there exists $a > 0$ such that $\|(T-z)x\| \geq a$ for $x \in S$, so that $\|(T-z)x\| \geq a\|x\|$ $(x \in X)$. From this it follows that the range Y of $T-zI$ is closed. $Y = X$ would imply that $T-z$ has a bounded inverse, so we must have $Y \neq X$. There therefore exists $f \in S'$ such that $f(Y) = \{0\}$. Then, for $x \in X$, $((T-z)^*f)x = f((T-z)x) = 0$. Hence $T^*f - zf = (T-z)^*f = 0$, and $z = [zf,f] = [T^*f,f] \in W(T^*)$.

The result $sp(T) \subset V(T^*)^-$ mentioned above may be proved directly, or will follow from theorem 1 together with the following result, which is due to Bonsall.

THEOREM 2.2. Let X be a normed space, and $T \in B(X)$. Then

$$V(T) \subset V(T^*).$$

Proof. Let $(x,f) \in P(X)$. Define $\hat{x} \in X''$ by $\hat{x}(f) = f(x)$ $(f \in X')$. Then it is easy to see that $(f,\hat{x}) \in P(X')$, and $\hat{x}(T^*f) = T^*f(x) = f(Tx)$. Since $f(Tx)$ is a general point of $V(T)$, we have $V(T) \subset V(T^*)$.

Now turning our attention to the numerical range $V(T)$ of an operator T on a complex space X, we have Williams's result that $sp(T) \subset V(T)^-$, and Zenger's result that $co\ sp(T) \subset V(T)$ for X a finite dimensional space. Since $V(T)$ need not be convex, the first fact does not imply the second. We now extend Zenger's result to a general Banach space. Professor Bonsall suggested the consideration of the approximate point spectrum.

THEOREM 2.3. Let X be a complex Banach space, and let $T \in B(X)$. Then $V(T)^-$ contains $co\ sp(T)$.

Proof. Firstly, it is clear that $co\ sp(T) = co\ \partial\, sp(T)$. Since $\partial\, sp(T) \subset \pi(T) \subset sp(T)$ by theorem 1.5, we have $co\ sp(T) = co\ \pi(T)$. Assume, without loss of generality, that $\|T\| = 1$. Suppose that $z \in co\ sp(T)$. Then there exists $z_k \in \pi(T)$ and $a_k \geq 0$ $(k = 1,2,3)$, such that $a_1+a_2+a_3 = 1$ and $a_1z_1+a_2z_2+a_3z_3 = z$. Assume first that $a_k > 0$ $(k = 1,2,3)$, and that the z_k are unequal. Suppose that $0 < \epsilon < r^2/24$, where $r = \min(|z_1 - z_2|, |z_2 - z_3|, |z_3 - z_1|)$. Since $z_k \in \pi(T)$, we can find $x_k \in S$ such that $\|(T-z_k)x_k\| < \epsilon$ $(k = 1,2,3)$. Let $Y = \mathrm{lin}\,(x_1,x_2,x_3)$. By Zenger [31], theorem 2, there exist $b_k \in C$ $(k = 1,2,3)$ and $f \in Y'$ such that $f(b_k x_k) = a_k$ $(k = 1,2,3)$, and $\|x\| = 1 = \|f\| = f(x)$, where $x = b_1x_1+b_2x_2+b_3x_3$. By the Hahn-Banach theorem, we can extend f to X without increasing its norm. Writing f for the extension, we have $(x,f) \in P(X)$. Now, since $\|(T-z_k)x_k\| < \epsilon$ $(k = 1,2,3)$,

$\|b_1z_1x_1 + b_2z_2x_2 + b_3z_3x_3 - T(b_1x_1 + b_2x_2 + b_3x_3)\| \leq M\epsilon$

where $M = |b_1| + |b_2| + |b_3|$. Hence

$$\|b_1z_1x_1 + b_2z_2x_2 + b_3z_3x_3\| \leq 1 + M\epsilon .$$

Similarly, using the fact that $\|(T^2 - z_k^2)x_k\| = \|(T+z_k)(T-z_k)x_k\| \leq 2\epsilon$, we have

$$\|b_1z_1^2x_1 + b_2z_2^2x_2 + b_3z_3^2x_3\| \leq 1 + 2M\epsilon .$$

From the identity

$$b_1(z_1-z_2)(z_1-z_3)x_1 = z_2z_3(b_1x_1 + \ldots + b_3x_3) -$$

$$(z_2+z_3)(b_1z_1x_1 + \ldots + b_3z_3x_3) + (b_1z_1^2x_1 +\ldots+ b_3z_3^2x_3),$$

and using the fact that $|z_k| \leq 1$, we have

$$|b_1|\, r^2 \leq |b_1(z_1-z_2)(z_1-z_3)| \leq 4 + 4M\epsilon .$$

Similarly, $|b_k|\, r^2 \leq 4 + 4M\epsilon$ $(k = 2,3)$. Hence $Mr^2 \leq 12 + 12M\epsilon \leq 12 + Mr^2/2$, and so $M \leq 24/r^2$. Finally

$$|f(Tx) - (a_1z_1 + \ldots + a_3z_3)| =$$

$$|b_1f(Tx_1 - z_1x_1) +\ldots+ b_3f(Tx_3 - z_3x_3)| \leq M\epsilon \leq 24\epsilon/r^2 .$$

Since $f(Tx) \in V(T)$ and ϵ is arbitrary, $z \in V(T)^{-}$.

In the case of $z = a_1 z_1 + a_2 z_2$, $a_1 + a_2 = 1, a_1 > 0$, $a_2 > 0$, $z_1 \neq z_2$, a similar argument will again show that $z \in V(T)^{-}$.

The proof of this theorem is independent of Williams's result that $sp(T) \subset V(T)^{-}$. Hence the latter may be regarded as a consequence of theorem 3.

We know that eigenvalues of T are contained in $V(T)$. The next theorem shows that eigenvalues of ascent greater than one are in fact interior points of $V(T)$. We shall require the following generalisation due to Kakutani [16], of Brouwer's fixed point theorem. We in fact give his corollary.

LEMMA 2.4. Let A be a bounded closed convex set in a Euclidean space. Let $K(A)$ be the family of all closed convex subsets of A. Then, if $x \to f(x)$ is an upper-semi-continuous point-to-set mapping of A into $K(A)$, then there exists an $x_0 \in A$ such that $x_0 \in f(x_0)$.

THEOREM 2.5. Let X be a complex normed space, and let $T \in B(X)$. Suppose that $u \in S$, and $\|T^2 u\| < \|Tu\|^2/8$. Then 0 is an interior point of $V(T)$. Given that also $T^2 u = 0 \neq Tu$, then $V(T)$ contains the disc $\{z : |z| \leq (3-8^{\frac{1}{2}})\|Tu\|\}$.

Proof. Write $R = \|Tu\|$ and $R' = \|T^2 u\|$. Suppose that z is any complex number with $|z| = r < R$. Let $x = zu + Tu$, and $y = x/\|x\|$. Let f be any element of $D(y)$. Then $\|x\| = f(x) = zf(u) + f(Tu)$, so that

$\|x\| f(Ty) = f(Tx) = zf(Tu) + f(T^2u) = z(\|x\| - zf(u)) + f(T^2u)$. Therefore $|f(Ty) - z| \leq |f(T^2u) - z^2 f(u)| / \|x\| \leq (r^2 + R')/(R - r)$. Define $F(z)$ as $V(T,y)$. Then we have

$$w \in F(z) \text{ implies } |w-z| \leq (r^2 + R')/(R - r). \qquad (1)$$

Since $R' < R^2/8$, there exists $c > 0$ such that

$$\frac{(\frac{1}{4}(R+a))^2 + R'}{R - \frac{1}{4}(R+a)} \leq \frac{R - 3a}{4} \qquad (0 \leq a \leq c) \qquad (2)$$

Suppose that $|t| = a \leq c$. Let D denote $\{z : |z| \leq \frac{1}{4}(R-3a)\}$. For $z \in D$, define $G(z)$ as $t + z - F(t + z)$. For $z \in D$ and $w \in F(t + z)$, since $|t + z| \leq \frac{1}{4}(R + a)$, from (1) and (2) we have $|t + z - w| \leq \frac{1}{4}(R - 3a)$. Hence $G(z)$ is a subset of D. From theorem 1.10 we know that the mapping $x \to V(T,x)$ is upper semi-continuous from S with the norm topology into convex compact subsets of C. Hence the mapping $z \to G(z)$ is upper semi-continuous from D into convex compact subsets of D. Lemma 4 now shows that there exists $z \in D$ such that $z \in G(z)$. Therefore $t \in F(t+z) \subset V(T)$. Since the argument goes through for any t with $|t| \leq c$, 0 is an interior point of $V(T)$.

In the case $T^2u = 0 \neq Tu$, we may apply the above argument with $R' = 0$. It is easily verified that we may take c to be $(3 - 8^{\frac{1}{2}})R$ in (2). Hence the disc $\{z : |z| \leq (3 - 8^{\frac{1}{2}}) \|Tu\|\}$ is contained in $V(T)$.

Theorem 5 will also apply if $8\|T^2\| < \|T\|^2$, for then $u \in S$ may be found such that $8\|T^2u\| < \|Tu\|^2$.

Given an operator T on a complex space, with an eigenvalue z of ascent greater than one, we can find $u \in S$ such that $(T-z)^2u = 0 \neq (T-z)u$. Application of theorem 5 with T replaced by $T-z$ now shows that $V(T)$ contains the disc $\{w : |w-z| \leq (3 - 8^{\frac{1}{2}})\|(T-z)u\|\}$.

Theorem 5 is also true for a real space X. The proof need only be modified by restricting all scalars to the real line. The remark following theorem 5 is also true in the real case, assuming z is real.

COROLLARY 2.6. Let X be a normed space with $T \in B(X)$, and let z belong to the boundary of $V(T)$. Then

$$8\|x\| \, \|(T-z)^2x\| \geq \|(T-z)x\|^2 \qquad (x \in X).$$

<u>Proof.</u> Suppose that $8\|x\| \, \|(T-z)^2x\| < \|(T-z)x\|^2$ for some $x \in X$. Put $u = x/\|x\|$. Then $u \in S$ and $8\|(T-z)^2u\| < \|(T-z)u\|^2$. By theorem 5, 0 is an interior point of $V(T-z)$. Since $V(T-z) = V(T)-z$, z is an interior point of $V(T)$, which is a contradiction.

We showed earlier that $V(T) \subset V(T^*)$ for any operator T. We now give an example in which $V(T) \neq V(T^*)$.

Let c_0, ℓ_1, ℓ_∞ denote respectively the Banach spaces of all complex sequences that are convergent to zero, that have absolutely convergent series, and that are bounded. We make the usual identifications $c_0' = \ell_1$, $\ell_1' = \ell_\infty$.

EXAMPLE 2.7. Let $X = c_0$, and define $T \in B(X)$ by

$$(Tx)_i = 2^{-1}x_i + 2^{-2}x_{i+1} + 2^{-3}x_{i+2} + \dots \qquad (x \in X).$$

Then $V(T) \neq V(T^*)$.

Proof. It is clear that $\|Tx\| < 1$ for $\|x\| = 1$. Therefore $z \in V(T)$ implies $|z| < 1$. We may write $X' = \ell_1$ and $X'' = \ell_\infty$. Then it is easily verified that

$$(T^*f)_i = 2^{-i}f_1 + 2^{-i+1}f_2 + \dots + 2^{-1}f_i \qquad (f \in X) .$$

Take $f = (1,0,0,0,\dots) \in X'$ and $\phi = (1,1,1,\dots) \in X''$. Then $(f,\phi) \in P(X')$, and $\phi(T^*f) = 1$. Hence $1 \in V(T^*)$, yet $1 \notin V(T)$.

It may be shown that, in this example, $V(T) = \{z : |z-\frac{1}{2}| < \frac{1}{2}\}$ while $V(T^*) = \{z : |z-\frac{1}{2}| \leq \frac{1}{2}\}$. We therefore have $V(T)^- = V(T^*)^-$ in this case. It is not known whether this always holds. However, it is always true that $\overline{co}\, V(T) = \overline{co}\, V(T^*)$. For, from theorem 1.9, for $|c| = 1$, $\sup\{\operatorname{Re} z : z \in V(cT^*)\} =$
$$\inf_{a>0} \frac{\|I^*+acT^*\| - 1}{a} = \inf_{a>0} \frac{\|I+acT\| - 1}{a} = \sup\{\operatorname{Re} z : z \in V(cT)\}.$$
Also, $v(T) = v(T^*)$.

Given a normed space X, the set $P(X)$ defined in chapter 1 as a subset of $X \times X'$ is known to be connected in the norm x weak* topology, but it is not known whether it is always connected in the norm x norm topology. The next theorem gives a case in which the latter is true.

THEOREM 2.8. Let E be any set, and let X be a normed linear space of functions on E, with the sup norm, with the property that, for $f,g \in S(X)$, there exist

$a, b \in E$ and $h \in S(X)$ with $1 = |f(a)| = |g(b)| = |h(a)| = |h(b)|$. Then P is (arcwise-) connected in the norm x norm topology of $X \times X'$.

Proof. Suppose that (f, ϕ) and $(g, \psi) \in P$. Define ϕ_a and $\phi_b \in X'$ by $\phi_a(k) = \overline{h(a)}\, k(a)$ $(k \in X)$ and $\phi_b(k) = \overline{h(b)}\, k(b)$ $(k \in X)$, where $a, b \in X$ and $h \in S(X)$ are such that $|f(a)| = |g(b)| = |h(a)| = |h(b)| = 1$. $(zf, \bar{z}\phi) \in P$ for $|z| = 1$, and hence these points are arcwise-connected. Choose z such that $zf(a) = h(a)$. Then $(zf, r\bar{z}\phi + (1-r)\phi_a) \in P$ for $0 \leq r \leq 1$, and $(rzf + (1-r)h, \phi_a) \in P$ $(0 \leq r \leq 1)$. Therefore (f, ϕ) is connected to (h, ϕ_a). Similarly, (g, ψ) is connected to (h, ϕ_b). Finally, since $(h, r\phi_a + (1-r)\phi_b) \in P$ $(0 \leq r \leq 1)$, we have (f, ϕ) connected to (g, ψ).

Let c, c_0 denote respectively the Banach spaces of all complex sequences that are convergent, and that are convergent to zero. Then each of these is a linear space of functions on the positive integers which has the property of theorem 8. Hence, in both cases, P is norm x norm connected.

§2. The joint numerical range.

Let X be a normed linear space, and $T_1, \ldots, T_n \in B(X)$. Then the joint numerical range $V(T_1, \ldots, T_n)$ of $T_1, \ldots, T_n$ is defined as $\{(f(T_1x), \ldots, f(T_nx)) : (x, f) \in P(X)\}$. Similarly, let A be a unital normed algebra, and $\underline{a} = (a_1, \ldots, a_n) \in A^n$. Then the joint numerical range $V(\underline{a})$ of $\underline{a}$ is defined as

$\{(f(a_1),\ldots,f(a_n)) : f \in D(1)\}$. Now let A be complete and over the complex field. The joint spectrum $sp(\underline{a})$ of $\underline{a}$ is the set of points $(z_1,\ldots,z_n)$ such that either

$$A(z_1-a_1) + \ldots + A(z_n-a_n) \neq A$$

or

$$(z_1-a_1)A + \ldots + (z_n-a_n)A \neq A .$$

In [5], Bonsall shows that, for $a \in A$, $sp(a) \subset V(a)$ and in [8] by a similar argument that $sp(\underline{a}) \subset V(\underline{a})$. Let N denote the set of all algebra norms p on A that are equivalent to the given norm, and satisfy $p(1) = 1$. Let $V_p(\underline{a})$ denote the numerical range of $\underline{a}$ with respect to the norm p. The following result is due to Bonsall [8].

THEOREM 2.9. Let A be a complex unital Banach algebra, and let $a_1,\ldots,a_n$ be mutually commuting elements of A. Then

$$\bigcap \{V_p(\underline{a}) : p \in N\} \subset \text{co } sp(a_1) \times \ldots \times \text{co } sp(a_n)$$

The latter set can be larger than co $sp(\underline{a})$.

THEOREM 2.10. With the same conditions as theorem 9,

$$\bigcap \{V_p(\underline{a}) : p \in N\} = \text{co } sp(\underline{a}) .$$

Proof. Since $sp(\underline{a}) \subset V_p(\underline{a})$, and $V_p(\underline{a})$ is convex, it is clear that

$$\bigcap \{V_p(\underline{a}) : p \in N\} \supset \text{co } sp(\underline{a}) . \qquad (1)$$

Let T be an nxn matrix, regarded as a linear mapping of A^n into A^n as well as of C^n into C^n.

For $\underline{b} = (b_1,\ldots,b_n) \in A^n$, it is easily verified that $V(T\underline{b}) = TV(\underline{b})$. Hence

$$\bigcap V_p(T\underline{b}) \supset T \bigcap V_p(\underline{b}).$$

If T is non-singular, we also have $\bigcap V_p(\underline{b}) \supset T^{-1} \bigcap V_p(T\underline{b})$, so that

$$\bigcap V_p(T\underline{b}) = T \bigcap V_p(\underline{b}).$$

By a similar argument, $sp(T\underline{a}) = T\, sp(\underline{a})$.

Suppose $\underline{w} = (w_1,\ldots,w_n) \in C^n \setminus co\ sp(\underline{a})$. Since $co\ sp(\underline{a})$ is convex and compact, there exists a linear functional f on C^n and a real number r such that

$$\text{Re } f(\underline{z}) < r < \text{Re } f(\underline{w}) \qquad (\underline{z} \in co\ sp(\underline{a})\ )$$

Let $f(\underline{z}) = t_{11}z_1 + \ldots + t_{1n}z_n \quad (\underline{z} = (z_1,\ldots,z_n) \in C^n)$, and let T be a non-singular nxn matrix with $(t_{11},\ldots,t_{1n})$ as its first row. Then

$$\text{Re } z_1 < r < \text{Re } x_1 \qquad (\underline{z} = (z_1,\ldots,z_n) \in sp(T\underline{a})\ )$$

where $(x_1,\ldots,x_n) = T\underline{w}$. Hence

$$T\underline{w} \notin co\ sp(c_1) \times \ldots \times co\ sp(c_n),$$

where $(c_1,\ldots,c_n) = T\underline{a}$, so that, by theorem 9, $T\underline{w} \notin \bigcap \{V_p(T\underline{a}) : p \in N\}$. This implies

$$\underline{w} \notin \bigcap \{V_p(\underline{a}) : p \in N\}.$$

This gives the opposite inclusion to (1), and so the result is proved.

Chapter 3

The numerical radius.

In this chapter, we study the numerical radius of an operator in relation to its norm. Recall that the numerical radius $v(T)$ of an operator T is defined as $\sup \{ |z| : z \in V(T) \}$.

§1. Complex spaces.

For an operator T on a complex Hilbert space, we have $\|T\| \leq 2w(T)$. J.P.Williams and T.Crimmins [30] have investigated the extreme case in which $\|T\| = 2w(T)$ and T attains its norm. They prove the following theorem.

THEOREM 3.1. Let X be a complex Hilbert space, and suppose that $w(T) = 1$ and $\|Tx\| = 2\|x\|$ for some $x \in X$. Then $T^2x = 0$, and lin (x, Tx) is a reducing subspace of T.

In the case of X a real Hilbert space, we can have $w(T) = 0$ and $T \neq 0$. We now prove a result analagous to theorem 1.

THEOREM 3.2. Let X be a real Hilbert space, and suppose that $v(T) = 0$ and $\|Tx\| = \|T\|$ for some $x \in S(X)$. Then $T^* = -T$, and lin (x, Tx) is a reducing subspace of T.

<u>Proof</u>. From the remark after theorem 2.7, we have $v(T^*) = v(T) = 0$. Hence $v(T + T^*) \leq v(T)+v(T^*) = 0$,

and since $T + T^*$ is self-adjoint, $T + T^* = 0$. Therefore $\|T\|^2 = (Tx,Tx) = (T^*Tx,x) = (-T^2x,x) \leq \|T^2\| \leq \|T\|^2$. Hence $(T^2x,x) = -\|T^2x\|\,\|x\|$, and, since we have equality in the Cauchy-Schwartz inequality, we must have $T^2x = -x$. It is now clear that $\mathrm{lin}(x,Tx)$ is invariant for both T and T^*, and so is a reducing subspace of T.

Given an operator T on a complex Hilbert space, Berger [2] has proved that $w(T^n) \leq w(T)^n$, and hence that $\|T^n\| \leq 2w(T)^n$. It was proved by Bohnenblust and Karlin [4] that, for an operator T on a complex normed space, $\|T\| \leq ev(T)$. This gives the estimate $\|T^n\| \leq e^n v(T)^n$ for a positive integer n. We now derive a much better estimate for T^n in terms of $v(T)$. Since, as we remarked after theorem 1.9, both the norm and numerical radius of an operator is unaltered on passing to the completion of a space, we may assume, without loss of generality, that X is complete.

THEOREM 3.3. Let X be a complex Banach space, and $T \in B(X)$. Then

$$\|T^n\| \leq n!(e/n)^n v(T)^n \leq en^{\frac{1}{2}}v(T)^n \qquad (n = 1,2,3,\ldots).$$

Proof. Assume first that $v(T) \leq 1$. Let m,n be positive integers with $m > n$. Let w_k $(k=1,2,\ldots,m)$ be the m th roots of unity. Then $\|\exp(zT)\| \leq \exp(|z|)$ $(z \in C)$ by theorem 1.9. Hence

$$\left\| \frac{(nT)^n}{n!} + \frac{(nT)^{n+m}}{(n+m)!} + \ldots \right\| = \left\| \frac{1}{m} \sum_{k=1}^{m} w_k^{-n} \exp(w_k nT) \right\|$$
$$\leq \exp(n).$$

Letting $m \to \infty$, we have $\|T^n\| \leq n!(e/n)^n$. Given that $v(T) = 0$, then $v(xT) = 0$, and so $\|xT\| \leq e$ $(x \in C)$, using the above with $n = 1$. Hence $T = 0$, and the inequalities certainly hold. Given $v(T) \neq 0$, the result is proved by applying the above inequality to $T/v(T)$. Finally, from Stirling's formula, we have $n!(e/n)^n \leq en^{\frac{1}{2}}$.

We do not know of any operator T such that $v(T) = 1$ and $\{T^n\}$ is unbounded. The next theorem shows that if T is an operator on a finite dimensional space, or, more generally, is a meromorphic operator (Taylor [25] Caradus [9]) then $\{T^n\}$ is bounded whenever $v(T) \leq 1$.

THEOREM 3.4. Let X be a complex normed linear space, and let T be a bounded linear meromorphic operator on X. Then $\|T^n\| = O(v(T)^n)$.

<u>Proof</u>. Assume first that $v(T) = 1$. Suppose $z \in sp(T)$ with $|z| = 1$. Then there exists an idempotent P such that $TP = PT$, $(zI-T)P$ is nilpotent, and $(zI-T)(I-P)$ is invertible in $(I-P)X$. Since z is a boundary point of $V(T)$, theorem 2.5 shows that $(zI-T)^2x = 0$ implies $(zI-T)x = 0$. It follows that $(zI-T)P = 0$. Therefore

$$T = TP + T(I-P) = zP + T(I-P).$$

Since the non-zero points of $sp(T)$ are isolated, T may therefore be written

$$T = \sum_{i=1}^{m} z_i P_i + R^*$$

where $|z_i| = 1$, $P_i^2 = P_i$, $P_iR = RP_i = 0$; $P_iP_j = 0$ $(i \neq j)$, and $\rho(R) < 1$. Then

$$T^n = \sum_{i=1}^{m} z_i^n P_i + R^n$$

so that $\{\|T^n\| : n = 1,2,\dots\}$ is bounded. The proof is now completed for a general operator T by applying the above to $T/v(T)$.

Glickfeld [14] has given an example of an operator on a two-dimensional complex space for which $\|T\| = ev(T)$. Hence the constant e in the inequality $\|T\| \leq ev(T)$ is best possible. In example 5 we describe such an operator in a different way from Glickfeld. We shall show later that any such example must be equivalent to that of Glickfeld.

THEOREM 3.5. Let A be the algebra of elements $b + cu$, with $b, c \in C$, where $u^2 = 0$. Define $f(x) = \exp(x)$ $(0 \leq x \leq 1)$, $f(x) = ex$ $(x \geq 1)$. Define p on A by

$$p(b + cu) = |b|\, f(|c/b|) \qquad (b \neq 0)$$
$$= |c|\, e \qquad (b = 0)$$

Then p is an algebra-norm on A, and $p(u) = ev(u)$.

Proof. Clearly, f is continuous and increasing. Since the derivative of f is increasing, f is convex. Also, $f(t)/t$ is a decreasing function of t, so that $f(ct) \leq cf(t)$ for $c \geq 1$. Finally, the derivative of $\log f(t)$ is decreasing, and $\log f(0) = 0$. So, for $0 < s < t$, $\log f(s+t) - \log f(t) \leq \log f(s)$, or $f(s+t) \leq f(s)f(t)$.

As f is continuous, p is continuous in b and c

at any point with $b \neq 0$. Since, for $|b| < |c|$, $p(b + cu) = e|c| = p(cu)$, p is also continuous at points with $b = 0$ and $c \neq 0$.

Suppose that $a_1 = b_1 + c_1u$ and $a_2 = b_2 + c_2u$ are any two elements of A, with none of $b_1, b_2, b_1 + b_2$ zero. Put $z_1 = c_1/b_1$ and $z_2 = c_2/b_2$. Then

$$p(a_1 + a_2) = p(b_1 + b_2 + (b_1z_1 + b_2z_2)u)$$

$$= |b_1 + b_2| f(b_1z_1 + b_2z_2 / |b_1 + b_2|)$$

$$\leq |b_1 + b_2| f(|b_1z_1|/|b_1 + b_2| + |b_2z_2|/|b_1 + b_2|)$$

$$\leq (|b_1| + |b_2|) f(|b_1z_1|/(|b_1| + |b_2|) + |b_2z_2|/(|b_1| + |b_2|))$$

$$\leq |b_1| f(|z_1|) + |b_2| f(|z_2|)$$

$$= p(a_1) + p(a_2).$$

Also,

$$p(a_1a_2) = p(b_1b_2(1 + z_1u)(1 + z_2u))$$

$$= p(b_1b_2(1 + (z_1 + z_2)u))$$

$$= |b_1b_2| f(|z_1 + z_2|)$$

$$\leq |b_1b_2| f(|z_1| + |z_2|)$$

$$\leq |b_1b_2| f(|z_1|) f(|z_2|)$$

$$= p(a_1)p(a_2)$$

By the continuity of p, these inequalities remain true when the restraints on a_1 and a_2 are removed. Clearly $p(za) = |z| p(a)$ $(z \in C)$, $p(a) = 0$ implies $a = 0$, and $p(1) = 1$. Hence p is an algebra-norm on A. Also, for $|z| \leq 1$, $\exp(zu) = 1 + zu$, and

$|z|^{-1} \log p(\exp(zu)) = 1$. Therefore, by theorem 1.9, $v(u) = 1$, and by definition, $p(u) = e$.

Regarding A as a normed linear space, define an operator T on A by $Ta = ua$ $(a \in A)$. Then, by the remarks in chapter 1, $\|T\| = \|u\| = e$, and $v(T) = v(u) = 1$.

In theorem 6, we derive a consequence of the case of equality for some n in theorem 3. However, we do not know whether such operators exist for $n > 1$.

THEOREM 3.6 Let X be a complex normed space, with $T \in B(X)$. $T \neq 0$. Suppose that $\|T^n\| = n!(e/n)^n v(T)^n$ for some positive integer n, and that $p(T) = 0$ for some non-zero complex polynomial p. Then 0 is an eigenvalue of T of ascent greater than n.

Proof. Assume, without loss of generality, that $v(T) = 1$, Then, by theorem 1.9, $\|\exp(zT)\| \leq \exp(|z|)$ $(z \in C)$. Suppose that $x \in S$ and $f \in S'$. Then

$$\sum_{k=0}^{\infty} |n^k f(T^k x)/k!|^2 =$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{k=0}^{\infty} \frac{\exp(kit)}{k!} n^k f(T^k x)\right)\left(\sum_{k=0}^{\infty} \frac{\exp(-kit)}{k!} \overline{n^k f(T^k x)}\right) dt$$

$$= (2\pi)^{-1} \int_0^{2\pi} |f(\exp(e^{it} nT)x)|^2 \, dt$$

$$\leq \exp(2n).$$

Since $\|T^n\| = n!(e/n)^n$, given any $\epsilon > 0$ we can choose x and f such that $|n^n f(T^n x)/n!|^2 >$ $\exp(2n) - \epsilon$. Then $|n^k f(T^k x)/k!| < \epsilon$ for $k \neq n$. Suppose that $a_0 I + \ldots + a_m T^m = 0$. Then

$a_o f(x) + a_1 f(Tx) + \ldots + a_m f(T^m x) = 0$, so that

$$|a_n f(T^n x)| \leq \epsilon \sum_{\substack{k=0 \\ k\neq n}}^{m} |a_k| k!/n^k.$$

Since ϵ is arbitrary, and $|f(T^n x)| > n!(e^{2n} - \epsilon)^{\frac{1}{2}}/n^n$, $a_n = 0$. Since also $a_o T^k + \ldots + a_{n-k} T^n + \ldots + a_m T^{m+k} = 0$ for $k = 1,2,\ldots,n$, we have $a_o = a_1 = \ldots = a_n = 0$.

Suppose that a_r is the non-zero coefficient of p with smallest suffix r, so that $r > n$. Then $a_r T^n + \ldots + a_n T^{m+n-r} \neq 0$, as the contrary would imply $a_r = 0$, by the above. Hence there exists $x \in X$ such that $(a_r T^n + \ldots + a_m T^{m+n-r})x \neq 0$. Let $y = (a_r I + \ldots + a_m T^{m-r})x$. Clearly, $T^n y \neq 0$ while $T^r y = 0$. This implies that 0 is an eigenvalue of T of ascent greater than n.

We now show that any operator T on a two-dimensional complex space, such that $\|T\| = ev(T)$ must behave as the operator of example 5. Indeed, we show that the existence of such an operator completely determines the geometry of the space.

THEOREM 3.7. Suppose that T is an operator on a two-dimensional complex space X, such that $v(T) = 1$ and $\|T\| = e$. Then $T^2 = 0$, and there exists $x \in S$ such that

$$\|I+zT\| = \|x+zTx\| = \exp(|z|) \qquad (|z| \leq 1)$$

$$\text{and} \quad \|I+zT\| = \|x+zTx\| = e|z| \qquad (|z| \geq 1).$$

<u>Proof</u>. T must satisfy a non-trivial quadratic equation which, by the proof of the previous theorem, must have coefficients of I and T zero. Therefore $T^2 = 0$.

We have

$$\|I+zT\| = \|\exp(zT)\| \leq \exp(|z|) \qquad (z \in C) \qquad (1)$$

Choose $x \in S$ and $f \in S'$ such that $f(Tx) = e$. The argument of the previous theorem shows that $f(x) = 0$. Therefore

$$\|(I+zT)x\| \geq |f((I+zT)x)| = e|z| \qquad (z \in C) \qquad (2)$$

From (1) and (2) we have

$$\|I+zT\| = \|x+zTx\| = e \qquad (|z| = 1) \qquad (3)$$

Let $|z| < 1$, and find $w \in C$ such that $|z+w| = |z| + |w| = 1$. Then

$$e = \|(I+(z+w)T)x\| = \|(I+wT)(I+zT)x\|$$
$$\leq \exp(|w|)\,\|x+zTx\|,$$

so that

$$\|(I+zT)x\| \geq \exp(|z|) \qquad (|z| < 1). \qquad (4)$$

From (1) and (4),

$$\|(I+zT)x\| = \|I+zT\| = \exp(|z|) \qquad (|z| < 1).$$

Now suppose that $|z| > 1$, and put $z = a+b$, with $|a| = 1$ and $|b| = |z| - 1$. Then, using (3) and the fact that $\|T\| = e$,

$$\|I+zT\| = \|(I+aT) + bT\| \leq e + (|z| -1)e =$$
$$|z|e \qquad (5)$$

Therefore, from (2) and (5),

$$\|I+zT\| = \|(I+zT)x\| = |z|e \qquad (|z| > 1).$$

It is clear that x and Tx are linearly independent, and so span X. Hence we have now found the norm of every element of X. The conditions of the theorem therefore determine completely the geometry of the space.

We showed in theorem 3 that, for an operator T on a complex space X, we have $\|T\| \leq ev(T)$. In the case of X uniformly convex, we can improve the constant e. For example, we now show that for a complex L_p space with $p \geq 2$, e may be replaced by $(e^p-1)^r$, where $r = 1/p$. In the case $p = 2$, the space is a Hilbert space, and we know that $\|T\| \leq 2v(T)$. The constant $(e^p-1)^r$ is not best possible in this case therefore, and possibly not for $p > 2$ either.

THEOREM 3.8. Let X be a complex L_p space with $p \geq 2$, and $T \in B(X)$. Then $\|T\| \leq (e^p-1)^r v(T)$, where $r = 1/p$.

Proof. Assume, without loss of generality, that $v(T)=1$. Let ϵ be any positive number less than one. Choose a positive integer n such that $\sum_{k=2n}^{\infty} \|T^k\|/k!$.

Let $w_k = \exp(k\pi i/n)$ $(k = 1,2,\ldots,2n)$. Let $x \in S$, and put $a_k = \|\exp(w_k T)x + \exp(-w_k T)x\|$ $(k =1,2,\ldots,2n)$. Then

$$\begin{aligned}\sum_{k=1}^{n} a_k &\geq \Big\|\sum_{k=1}^{n} \exp(w_k T)x + \exp(-w_k T)x\Big\| \\ &= \Big\|\sum_{k=1}^{2n} \exp(w_k T)x\Big\| \\ &= 2n\|x + T^{2n}x/(2n)! + T^{4n}x/(4n)! + \ldots\| \\ &> 2n(1 - \epsilon).\end{aligned}$$

By Clarkson's inequality [10], and theorem 1.9

$$\|\exp(w_k T)x - \exp(-w_k T)x\|^p + \|\exp(w_k T)x + \exp(-w_k T)x\|^p$$

$$\leq \quad 2^{p-1}(\|\exp(w_k T)x\|^p + \|\exp(-w_k T)x\|^p) \quad \leq \quad (2e)^p.$$

Using the definition of a_k, this gives

$$\| \exp(w_k T)x - \exp(-w_k T)x \| \leq ((2e)^p - a_k{}^p)^r.$$

Now

$$(2n)^{-1} \sum_{k=1}^{n} w_k{}^{-1}(\exp(w_k T)x - \exp(-w_k T)x)$$

$$= (2n)^{-1} \sum_{k=1}^{2n} w_k{}^{-1} \exp(w_k T)x$$

$$= Tx + T^{2n+1}x/(2n+1)! + \ldots$$

Therefore

$$\| Tx \| \leq (2n)^{-1} \sum_{k=1}^{n} ((2e)^p - a_k{}^p)^r + \epsilon.$$

The expression on the right hand side, under the conditions $0 \leq a_k \leq 2e$ and $\sum_{k=1}^{n} a_k \geq 2n(1 - \epsilon)$, attains its maximum when $a_k = 2(1 - \epsilon)$ $(k = 1,2,\ldots,n)$. To see this, consider the function f defined by $f(t) = (a-t^p)^r + (a-(b-t)^p)^r$, where a and b are positive constants. Then $f'(t) = (b-t)^{p-1}(a-(b-t)^p)^{r-1} - t^{p-1}(a-t^p)^{r-1}$ and $f'(t) > 0$ for $t < \frac{1}{2}b$, $f'(t) < 0$ for $t > \frac{1}{2}b$. Hence f is a maximum when $t = b-t = \frac{1}{2}b$. This shows that

$g(a_1,\ldots,a_n) = \sum_{k=1}^{n} ((2e)^p - a_k{}^p)^r$ is increased when a_i and a_j are each replaced by $\frac{1}{2}(a_i + a_j)$. Repeating this argument, and using the continuity of g, we have that g is increased when $a_1,\ldots,a_n$ are each replaced by $(a_1 + \ldots + a_n)/n$. Therefore

$$\| Tx \| \leq (e^p - (1 - \epsilon)^p)^r + \epsilon.$$

Since ϵ is an arbitrary positive number less than one, and x is an arbitrary element of S, we have

$$\|T\| \leq (e^p - 1)^r.$$

<u>Remark</u>. In the case of X a complex L_p space with $1 < p < 2$, a similar argument shows that $\|T\| \leq (e^q - 1)^{1/q} v(T)$ where $p^{-1} + q^{-1} = 1$.

Let E be a compact Hausdorff space, and X the Banach space of all continuous complex valued functions on E, with the sup norm. Then Duncan, McGregor, Pryce and White [13] have proved that $\|T\| = v(T)$ for every $T \in B(X)$. We now show that this result remains true if the topological space E is only assumed to be completely regular. *

THEOREM 5.9. Let E be a completely regular topological space, and X the space of all bounded continuous complex valued functions on E, with the sup norm. Let $T \in B(X)$. Then $\|T\| = v(T)$.

<u>Proof</u>. Given any $\epsilon > 0$, there exists $f \in S$ such that $\|Tf\| > \|T\| - \frac{1}{2}\epsilon$. Then there exists $a \in E$ such that $|Tf(a)| > \|T\| - \frac{1}{2}\epsilon$. If $f(a) = 0$, put $h = (f+c)/\|f+c\|$, where $c > 0$ is such that $\|Tf - Th\| < \frac{1}{2}\epsilon$. If $f(a) \neq 0$, put $h = f$. In either case, we have $h \in S$, $h(a) \neq 0$, and $|Th(a)| > \|T\| - \epsilon$. Let $U = \{ b \in E : |h(b)| > \frac{1}{2}|h(a)| \}$, so that U is an open neighbourhood of a. Since E is completely regular there exists $g \in X$ such that $0 \leq g(b) \leq 1$ $(b \in E)$, $g(b) = 0$ $(b \in E \setminus U)$, and $g(a) = 1$.

Define $k \in X$ by

* Taking the Stone-Cech compactification $\tilde{E}$ of E, we have $X \cong C(\tilde{E})$, and the result follows from [13].

$$k(b) = g(b)h(b)(1-|h(b)|^2)^{\frac{1}{2}}/|h(b)| \qquad (b \in U)$$
$$k(b) = 0 \qquad (b \in E \setminus U).$$

Then define $f_1, f_2 \in X$ by $f_1 = h + ik$ and $f_2 = h - ik$. For $b \in E \setminus U$, $|f_1(b)| = |h(b)| \leq 1$. For $b \in U$,

$f_1(b) = h(b) + ig(b)h(b)(1 - |h(b)|^2)^{\frac{1}{2}}/|h(b)|$, so that

$|f_1(b)|^2 = |h(b)|^2 + g(b)^2(1 - |h(b)|^2) \leq 1$. Since $g(a) = 1$, we have $|f_1(a)| = 1$, so that $\|f_1\| = 1$. Similarly, $f_2 \in S$ and $|f_2(a)| = 1$. Clearly, $f_1 + f_2 = 2h$, so that $|Tf_1(a) + Tf_2(a)| = 2|Th(a)|$. One of $|Tf_1(a)| \geq |Th(a)|$, $|Tf_2(a)| \geq |Th(a)|$ must be true. Assume the former. Let $\phi \in X'$ be defined by $\phi(f) = \overline{f_1(a)}\, f(a)$ $(f \in X)$. Then $(f_1, \phi) \in P$, and so $v(T) \geq |\phi(Tf_1)| = |Tf_1(a)| \geq |Th(a)| > \|T\| - \epsilon$. The same result will follow on the assumption $|Tf_2(a)| \geq |Th(a)|$. Since ϵ is arbitrary we have $v(T) = \|T\|$.

§2. Real spaces.

We have seen that when T is an operator on a complex normed space, $\|T\| \leq ev(T)$. No such inequality can hold when X is a real normed space, for then we can have $v(T) = 0$ and yet $T \neq 0$. For example, for $X = R^2$ regarded as a real Hilbert space, define T by $T(x,y) = (-y,x)$. However, Bonsall and Duncan [7] showed that $v(T) = v(T^2) = 0$ implies $T = 0$. It is natural to ask whether an inequality of the form $\|T\| \leq av(T) + bv(T^2)^{\frac{1}{2}}$ holds, where a and b are real constants. Theorem 10 shows that such a result does hold for real Hilbert space.

THEOREM 3.10. Let X be a real Hilbert space, and let $T \in B(X)$. Then

$$\|T\|^2 \leq 20v(T)^2 + 2v(T^2).$$

Proof. Write $a = v(T)$ and $b = v(T^2)$. Let $x \in S$. Then

$$\|Tx\|^2 = (x+Tx, Tx+T^2x) - (x, Tx) - (x, T^2x) - (Tx, T^2x)$$

$$\leq a(\|x\|^2 + \|Tx\|^2 + \|x+Tx\|^2) + b\|x\|^2$$

$$\leq 2a(1 + a + \|T\|^2) + b$$

where we have used $\|x+Tx\|^2 = \|x\|^2 + 2(Tx,x) + \|Tx\|^2 \leq 1 + 2a + \|T\|^2$. Since $x \in S$ is arbitrary, we have $\|T\|^2 \leq 2a(1 + a + \|T\|^2) + b$, or $(1-2a)\|T\|^2 \leq 2a(1+a) + b$. Now apply this result to rT, where $r > 0$. Since $v(rT) = rv(T)$, and $v((rT)^2) = r^2v(T^2)$, we must replace a and b by ra and r^2b respectively. This gives

$$(1 - 2ar)r^2\|T\|^2 \leq 2ar(1 + ar) + br^2.$$

Putting $r = (4a)^{-1}$, we deduce

$$\|T\|^2 \leq 20a^2 + 2b.$$

After I had proved this result, Lumer (unpublished) proved the following stronger inequality.

THEOREM 3.11. Let X be a real Hilbert space, and let

$T \in B(X)$. Then

$$\|T\| \leq 2w(T) + w(T^2)^{\frac{1}{2}}$$

and this inequality is sharp.

Proof. Let $x, y \in X$. Then

$$\begin{aligned} 2|(Tx,y) + (Ty,x)| &= |(T(x+y),x+y) - (T(x-y),x-y)| \\ &\leq w(T)(\|x+y\|^2 + \|x-y\|^2) \\ &= 2w(T)(\|x\|^2 + \|y\|^2). \end{aligned}$$

Therefore

$$|(Tx,y) + (Ty,x)| \leq w(T)(\|x\|^2 + \|y\|^2).$$

Assume that $T \neq 0$, and apply this to $x \in S$ and $y = Tx/\|T\|$. This gives $|\,\|Tx\|^2 + (T^2x,x)| \leq 2w(T)\|T\|$, which implies $\|T\|^2 \leq 2w(T)\|T\| + w(T^2)$. Hence $\|T\|$ satisfies

$$\|T\|^2 - 2w(T)\|T\| - w(T^2) \leq 0.$$

We must have $\|T\|$ not greater than the largest zero of the polynomial $t^2 - 2w(T)t - w(T^2)$, i.e.

$$\|T\| \leq w(T) + (w(T^2) + w(T)^2)^{\frac{1}{2}}.$$

Since $t^{\frac{1}{2}}$ is a subadditive function of t for $t \geq 0$, we have

$$\|T\| \leq 2w(T) + w(T^2)^{\frac{1}{2}}$$

To show that this is sharp, consider the operator T defined on R^2 by $T(x,y) = (-y,x)$. Then $w(T) = 0$ and $\|T\| = w(T^2)^{\frac{1}{2}}$.

Lumer has also shown that an inequality of the form $\|T\| \leq av(T) + bv(T^2)^{\frac{1}{2}}$, with a and b real constants, must hold for any operator on a real normed space. In theorem 12, we adapt his argument to give actual constants in the inequality. The theorem is stated for an element of a real Banach algebra, but, by the remarks in §1.3 will also apply to an operator on a real normed space.

THEOREM 3.12. Let A be a real Banach algebra with identity of norm one, and let $a \in A$. Then

$$\|a\| \leq \max(48v(a), 24v(a^2)^{\frac{1}{2}})$$

<u>Proof</u>. Let B be the complexification of A, so that A is embedded isometrically in B (Rickart [23] §1.3). Let b be the element in B corresponding to $a \in A$. Then

$$\begin{aligned} \sup\{|\mathrm{Re}\, z| : z \in V(b)\} &= \sup_{t\in R} \frac{\log\|\exp(tb)\|}{|t|} \\ &= \sup_{t\in R} \frac{\log\|\exp(ta)\|}{|t|} \\ &= \sup\{|\mathrm{Re}\, z| : z \in V(a)\} \\ &= v(a). \end{aligned}$$

Similarly, $\sup\{|\mathrm{Re}\, z| : z \in V(b^2)\} = v(a^2)$. Let $r > 0$ be such that

$$4v(a)r \leq 1 \quad \text{and} \quad 4v(a^2)r^2 \leq 1. \tag{1}$$

Let $z = re^{it}$, where $t \in R$. Let $c \in B$, with $\|c\| = 1$, and let $f \in D(c)$. Then $\|(1+zb)c\| \geq |f((1+zb)c)| = |1 + zf(bc)| = |e^{-it} + rf(bc)| \geq \mathrm{Re}(e^{-it} + rf(bc)) \geq \cos(t) - \frac{1}{4}$, since $f(bc) \in V(b)$, and using (1). It follows from this that, for $-\pi/3 \leq t \leq \pi/3$, $0 \notin V(1+zb)^-$ so that $1+zb$ is invertible, by theorem 1.6. We also have

$$\|(1+zb)^{-1}\| \leq (\cos(t) - \tfrac{1}{4})^{-1} \qquad (-\pi/3 \leq t \leq \pi/3)$$

Similarly, since $4v(a^2)r^2 \leq 1$, $(1+z^2b^2)$ is invertible for $-\pi/6 \leq t \leq \pi/6$, and

$$\|(1+z^2b^2)^{-1}\| \leq (\cos(2t) - \tfrac{1}{4})^{-1} \qquad (-\pi/6 \leq t \leq \pi/6).$$

This implies that, for $-\pi/6 \leq t \leq \pi/6$, $1+zib$ is invertible, and

$$\begin{aligned}\|(1+zib)^{-1}\| &= \|(1+z^2b^2)^{-1}(1+zib)\| \\ &\leq (\cos(2t) - \tfrac{1}{4})^{-1}(1 + r\|b\|).\end{aligned}$$

We now have that $(1 + re^{it}b)$ is invertible for $-\pi/3 \leq t \leq 2\pi/3$. The same arguments applied to $-b$ will show that $(1 + re^{it}b)$ is invertible for $-\pi/3 \leq t \leq 2\pi/3$. Hence we have $(1-zb)$ invertible for $|z| = r$, and also, by (1), for $|z| \leq r$. Therefore, for any positive integer n,

$$\begin{aligned}2\pi\|b^n\| &= \left\| \int_{|z|=r} z^{-n-1}(1-zb)^{-1}\,dz \right\| \\ &\leq r^{-n}\int_0^{2\pi} \|(1 - re^{it}b)^{-1}\|\,dt\end{aligned}$$

$$= r^{-n}\Big(\int_{-\pi/3}^{\pi/3} \|(1-re^{it}b)^{-1}\|\,dt + \int_{-\pi/6}^{\pi/6} \|(1-rie^{it}b)^{-1}\|\,dt$$

$$+ \int_{-\pi/3}^{\pi/3} \|(1+re^{it}b)^{-1}\|\,dt + \int_{-\pi/6}^{\pi/6} \|(1+rie^{it}b)^{-1}\|\,dt\Big)$$

$$\leq r^{-n}\Big(4\int_0^{\pi/3} (\cos(t) - \tfrac{1}{4})^{-1}\, dt \; +$$

$$4(1 + r\|b\|) \int_0^{\pi/6} (\cos(2t) - \tfrac{1}{4})^{-1}\, dt \Big)$$

$$= 2r^{-n}(3 + r\|b\|) \int_0^{\pi/3} (\cos(t) - \tfrac{1}{4})^{-1}\, dt$$

$$\leq 4r^{-n}(3 + r\|b\|)$$

since $\int_0^{\pi/3} (\cos(t) - \tfrac{1}{4})^{-1}\, dt = 2\int_0^{\pi/6} (\cos(2t) - \tfrac{1}{4})^{-1} dt$ < 2. Assume for the moment that $\|a\| = \|b\| = 1$. We want to establish an upper bound for r, so assume that $r > 1$. For $1 < t < r$, let $c = (1-tb)/\|1-tb\|$, and let $f \in D(c)$. Then

$$\|(1-tb)^{-1}c\| \geq |f((1-tb)^{-1}c)|$$

$$= |f(c + tbc + t^2b^2c + \ldots)|$$

$$\geq 1 - tv(a) - t^2v(a^2) - 2t^3(\pi r^3)^{-1}(3+r\|b\|) - \ldots$$

$$\geq 1 - t/(4r) - t^2/4r^2 - 2\pi^{-1}(3+r)t^3r^{-2}(r-t)^{-1}.$$

But we also have $\|(1-tb)^{-1}c\| = \|1-tb\|^{-1} \leq (t-1)^{-1}$. Hence

$$(t-1)^{-1} \geq 1 - t/(4r) - t^2/(4r^2)$$
$$- \frac{2}{\pi}(3+r)t^3r^{-2}(r-t)^{-1} \qquad (1 < t < r) \qquad (2)$$

This inequality is not satisfied for $t = 3$ and $r \geq 12$. We must therefore have $r < 12$.

First consider the case $v(a) = v(a^2) = 0$. Assume that $a \neq 0$. By replacing a by $a/\|a\|$, we may assume that $\|a\| = 1$. But $r = 12$ satisfies (1), and this contradicts (2). Hence $a = 0$, and the theorem holds in this case.

Now assume that $v(a)$ and $v(a^2)$ are not both zero, and that $\|a\| = 1$. Then $r = \min((4v(a))^{-1}, (2v(a^2)^{\frac{1}{2}})^{-1})$ satisfies (1), is therefore less than 12. Hence $\max(48v(a), 24v(a^2)^{\frac{1}{2}}) > 1$. Now remove the assumption that $\|a\| = 1$, and apply this to $a/\|a\|$. This gives $\max(48v(a)/\|a\|, 24v(a^2)^{\frac{1}{2}}/\|a\|) > 1$, or

$$\|a\| \leq \max(48v(a), 24v(a^2)^{\frac{1}{2}}).$$

Chapter 4

Hermitian Operators.

We now study bounded linear operators on a complex space which have real numerical range. Such operators are said to be Hermitian. In the case of a complex Hilbert space, a bounded operator is Hermitian in the above sense if and only if it is self-adjoint. Hermitian operators have many of the properties of self-adjoint operators.

Similarly, an element of a complex normed algebra is said to be Hermitian if it has real numerical range. Vidav [28] considered "self-adjoint" elements of a complex algebra, which were defined as elements $a \in A$ such that $\|1+ita\| = 1 + o(t)$ $(t \in R, t \to 0)$. Lumer [17] showed that these are precisely the elements with real numerical range, or Hermitian elements.

Throughout this chapter, X will always denote a complex normed linear space, and T will denote a bounded linear operator on X.

DEFINITION 4.1. T is said to be Hermitian if $V(T) \subset R$.

The following theorem gives alternative definitions of a Hermitian operator.

THEOREM 4.2. The following are equivalent

(i) T is Hermitian

(ii) $\| \exp(ixT) \| = 1 \quad (x \in R)$

(iii) $\| I + iaT \| = 1 + o(a) \quad (a \in R, a \to 0)$

Proof. We have

$$\sup \{ \operatorname{Im} z : z \in V(T) \} = \sup \{ \operatorname{Re} z : z \in V(-iT) \} = \lim_{a \to 0+} \frac{\| I - iaT \| - 1}{a}.$$ Similarly, $\sup \{ -\operatorname{Im} z : z \in V(T) \} =$

$\lim_{a \to 0+} \frac{\| I + iaT \| - 1}{a}$. Clearly, T is Hermitian if and only if $\lim_{a \to 0} \frac{\| I + iaT \| - 1}{a} = 0$, and this is equivalent to $\| I + iaT \| = 1 + o(a)$ as $a \to 0$ and $a \in R$.

For the third part, we have

$$\sup \{ |\operatorname{Im} z| : z \in V(T) \} = \sup \{ |\operatorname{Re} z| : z \in V(iT) \} =$$

$$\sup_{a \in R} \frac{\log \| \exp(iaT) \|}{|a|}$$

From this it is clear that T is Hermitian if and only if $\| \exp(iaT) \| \leq 1 \quad (a \in R)$. Since $1 = \| I \| =$ $\| \exp(iaT)\exp(-iaT) \| \leq \| \exp(iaT) \| \, \| \exp(-iaT) \|$, it follows that $\| \exp(iaT) \| \leq 1 \quad (a \in R)$ is equivalent to $\| \exp(iaT) \| = 1 \quad (a \in R)$.

The following basic facts about Hermitian operators were proved by Vidav.

THEOREM 4.3. Suppose that T and U are Hermitian operators on a normed space X. Then

(i) aT is Hermitian $(a \in R)$.

(ii) $T + U$ is Hermitian.

(iii) $i(TU - UT)$ is Hermitian.

(iv) $\|\exp(iaT)V\| = \|V\|$ for any operator V on X
$(a \in R)$.

(v) $sp(T) \subset R$.

For a Hermitian operator T on a complex Hilbert space H, we have, for $x \in H$, $(T^2x,x) = (Tx,Tx)$, and so $\|Tx\|^2 \leq \|x\| \; \|T^2x\|$. In the case of a Hermitian operator on a normed space X, since $V(T) \subset R$, 0 is not an interior point of $V(T)$. Hence, by theorem 2.5, we have, for $x \in X$, $\|Tx\|^2 \leq 8\|x\| \; \|T^2x\|$. Bonsall has shown how the constant 8 may be improved in this inequality. The proof is given in Bonsall and Duncan [8].

THEOREM 4.4. Suppose that T is a Hermitian operator on a normed space X. Then

$$\|Tx\|^2 \leq 4\|x\| \; \|T^2x\| \quad (x \in X).$$

An operator T is said to be dissipative if $\operatorname{Re} z \leq 0$ for $z \in V(T)$. Bohnenblust and Karlin [4] made a conjecture which is equivalent to: a dissipative operator T with $sp(T) = \{0\}$ must be the zero operator. Lumer and Phillips [19] provided a counterexample to this. They prove, however, that the conjecture is true under the additional assumption that T satisfies a non-trivial polynomial equation. We now prove a result which may be regarded as an extension of their result: given that T is dissipative, $sp(T)$ is contained in the imaginary axis, and T satisfies a non-trivial polynomial equation, then $V(T)$ is contained in the imaginary axis. The theorem is stated in a slightly different form.

THEOREM 4.5. Let X be a Banach space, and $T \in B(X)$. Suppose that $\operatorname{Im} z \geq 0$ $(z \in V(T))$, and that $p(T) = 0$ for some non-zero complex polynomial p, which has all its zeros real. Then T is Hermitian.

<u>Proof</u>. The eigenvalues of T are real, and so lie on the boundary of $V(T)$. By theorem 2.5, they have ascent one, and we may assume that p has simple zeros. Suppose that $p(z) = a_0 + \ldots + a_k z^k$, with unequal roots $z_1, \ldots, z_k$. Operators $P_1, \ldots, P_k$ may be found so that $T^r = z_1^r P_1 + \ldots + z_k^r P_k$ for $r = 0, 1, \ldots, k-1$. Then the recurrence relation for the powers of T shows that $T^n = z_1^n P_1 + \ldots + z_k^n P_k$ for $n = k, k+1, \ldots$ also. Hence, $\exp(zT) = \exp(z_1 z)P_1 + \ldots + \exp(z_k z)P_k$ $(z \in C)$.

Suppose that $0 < t < 1$ and $\epsilon > 0$. Let $w_p = \exp(iz_p)$ $(p = 1, \ldots, k)$. Consider the sequence $\{(w_1^n, \ldots, w_k^n) : n = 1, 2, \ldots\}$. By the compactness of D^k, where $D = \{z : |z| = 1\}$, there exist positive integers m, n with $m - n = q > 0$ such that $|w_p^m - w_p^n| < \epsilon$, and so $|w_p^q - 1| < \epsilon$, for $p = 1, \ldots, k$. Therefore $|e^{-itz_p} - e^{i(q-t)z_p}| = |w_p^q - 1| < \epsilon$ $(p = 1, \ldots, k)$, so that $\| e^{-itT} - e^{i(q-t)T} \| \leq \epsilon (\|P_1\| + \ldots + \|P_k\|) = \epsilon K$.

Now $\sup_{a>0} \frac{\log \|e^{iaT}\|}{a} = \sup \operatorname{Re} V(iT) = 0$. Hence $\|e^{iaT}\| \leq 1$ for $a > 0$. Therefore $\|e^{-itT}\| \leq 1 + \epsilon K$, since $q-t > 0$. Since ϵ is arbitrary, $\|e^{-itT}\| \leq 1$. Hence $\sup \operatorname{Re} V(-iT) = \lim_{t \to 0+} t^{-1} \log \|e^{-itT}\| \leq 0$.

Together with $\sup \operatorname{Re} V(iT) = 0$, this implies that $V(T) \subset R$, or T is Hermitian.

The next result is due to Vidav, and is known as Vidav's lemma. In the proof given below, the Phragmen-Lindelof theorem which Vidav used is established in an elementary way. This simplification is due to Lumer and Bonsall.

THEOREM 4.6. Given that T is Hermitian, we have $v(T) = \rho(T)$.

Proof. Let $a = \sup \{z : z \in sp(T)\}$. The function $f(t) = \log \|e^{tT}\|$ is continuous and subadditive for $t > 0$. Therefore $\inf_{t>0} t^{-1}\log \|e^{tT}\| = \lim_{t \to \infty} t^{-1}\log \|e^{tT}\|$ $= \lim_{n \to \infty} \log(\|e^{nT}\|^{1/n}) = \log(\rho(e^T)) = a$. This implies that $\|e^{tT}\| \geq e^{at}$ $(t > 0)$. Let $\epsilon > 0$. For $z \in C$ with $\operatorname{Re} z \geq 0$, define $f(z) = e^{z(T-a-\epsilon)}$ and $g(z) = f(z)/(1+\epsilon z)$. Then $f(z) = \|e^{x(T-a-\epsilon)}\| \to 0$ as $x \to \infty$, where $x = \operatorname{Re} z$. Hence there exists M such that $\|f(z)\| \leq M$ for $\operatorname{Re} z \geq 0$.

We have $\|g(z)\| \leq 1$ for $\operatorname{Re} z \geq 0$ and $|z| = R$, where $R \geq (M+1)/\epsilon$. Also, $\|g(z)\| \leq \|f(z)\| = 1$ for $\operatorname{Re} z = 0$. The maximum modulus theorem now shows that $\|g(x)\| \leq 1$ for $x > 0$, or $\|e^{x(T-a-\epsilon)}\| \leq 1 + \epsilon x$ for $x > 0$. Letting $\epsilon \to 0$, we deduce $\|e^{x(T-a)}\| \leq 1$, or $\|e^{xT}\| \leq e^{xa}$ $(x > 0)$. Together with the opposite inequality proved above, this gives $\|e^{xT}\| = e^{xa}$ $(x > 0)$. Finally

$$\sup \{ \operatorname{Re} z : z \in V(T) \} = \sup_{x>0} x^{-1} \log \|e^{xT}\| = a$$

$$= \sup \{ \operatorname{Re} z : z \in sp(T) \} .$$

The same argument applied to $-T$ will show that

$$\inf \{ \operatorname{Re} z : z \in V(T) \} = \inf \{ \operatorname{Re} z : z \in sp(T) \}.$$

Therefore, $v(T) = \rho(T)$

For a Hermitian operator T, we have $\|T\| \leq e\,\rho(T)$, since $\rho(T) = v(T)$. Using entire function theory, A.M. Sinclair [24] has recently established the result that $\|T + zI\| = \rho(T + zI)$ for T Hermitian and $z \in C$. We shall require the following lemma, proved by Lumer and Phillips [19].

LEMMA 4.7. Let A be a complex Banach algebra, and let $a \in A$, with $\rho(a) < r$. Then there exists a constant M such that

$$\|e^{za}\| \leq Me^{r|z|} \quad (z \in C).$$

Proof. Choose t such that $\rho(a) < t < r$. There exists an integer N such that $\|a^n\|^{1/n} < t$ for $n > N$. Hence

$$\|e^{za}\| \leq \sum_{n=1}^{N} |z|^n \|a\|^n/n! + e^{t|z|} \leq e^{r|z|}$$

for $|z|$ sufficiently large. Therefore, there exists a constant $M > 0$ such that $\|e^{za}\| \leq Me^{r|z|}$ $(z \in C)$.

The conclusion of lemma 7 implies that the entire function e^{za} is of exponential type at most $\rho(a)$. We are now ready to prove Sinclair's result.

THEOREM 4.8. Let A be a complex Banach algebra with identity of norm one. Let $h \in H$ be Hermitian. Then

$\|h + z\| = \rho(h + z)$ $(z \in C)$.

Proof. Firstly, if $z = x + iy$, with x and y real, then $h + z = (h+x) + iy$ and $h+x$ is Hermitian. Hence it is sufficient to prove the result for z purely imaginary.

Let f be a bounded linear functional on A of norm one. Then $f(\exp(zih))$ is an entire function of exponential type at most $\rho(h)$ (by lemma 7) whose modulus is bounded by one for all real z. We now require a generalisation of a theorem of S. Bernstein [3]. If $F(z)$ is an entire function of exponential type t whose modulus is bounded by 1 for all real z, then $|F'(z) - cF(z)| \leq (t^2 + c^2)^{\frac{1}{2}}$ $(t, c \in R)$.

Apply this with $F(z) = f(\exp(zih))$ and $z = 0$. Therefore

$$|f(h) + icf(1)| \leq |\rho(h) + ic| .$$

Since $sp(h) \subset R$, $\rho(h + ic) = |\rho(h) + ic|$ $(c \in R)$. Hence, since $\|h + ic\| = \sup\{|f(h + ic)| : f \in A', \|f\| = 1\}$, $\|h + ic\| = \rho(h + ic)$.

We shall show from the next theorem that the norm of a Hermitian operator is equal to its spectral radius. The proof requires the following application of the Phragmen-Lindelof principle, which is due to F. Carlson (Titchmarsh [26] theorem 5.8.1).

LEMMA 4.9. If $f(z)$ is regular and of the form $O(e^{k|z|})$ where $k < \pi$ for $\operatorname{Re} z \geq 0$, and $f(z) = 0$ for $z = 0, 1, 2, \ldots$, then $f(z) = 0$ for $\operatorname{Re} z \geq 0$.

THEOREM 4.10. Let A be a complex Banach algebra with identity of norm one. Let H be the set of Hermitian elements of A. Then, for $h \in H$ with $\rho(h) \leq \pi/2$, we have

$$h = \frac{4}{\pi}\left(\sin(h) - \sin(3h)/3^2 + \sin(5h)/5^2 - \ldots \right)$$

$$h^2 = \pi^2/12 - \cos(2h) + \cos(4h)/2^2 - \cos(6h)/3^2 + \ldots$$

Proof. For $h \in H$, define $f(h) \in A$ by

$$f(h) = \frac{4}{\pi}\left(\sin(h) - \sin(3h)/3^2 + \sin(5h)/5^2 - \ldots\right)$$

$$= \frac{2}{i\pi}\left(\ldots + e^{-3ih}/3^2 - e^{-ih} + e^{ih} - e^{3ih}/3^2 + \ldots\right)$$

$$= \sum_{n=-\infty}^{\infty} a_n e^{nih} \qquad (1)$$

We have $\sum_{n=-\infty}^{\infty} |a_n| = \pi/2 = M$ (say), so that the series is absolutely convergent, and $f(h)$ is well defined. In forming powers of $f(h)$, we may therefore multiply term by term, and rearrange the product terms without altering the sum. Hence

$$f(h)^n = \sum_{m=-\infty}^{\infty} a_{n,m} e^{mih} \qquad (n = 1,2,\ldots) \qquad (2)$$

where $a_{n,m} = \sum \{ a_{i_1} a_{i_2} \dots a_{i_n} : i_1 + i_2 + \dots + i_n = m \}$.

Clearly

$$\sum_{m=-\infty}^{\infty} |a_{n,m}| \leq M^n \qquad (n = 1,2,\dots) \qquad (3)$$

Let $z \in C$. Then $\sin(zf(h)) =$

$\sum_{n=1}^{\infty} \frac{1}{2}i((-iz)^n - (iz)^n)/n! \sum_{m=-\infty}^{\infty} a_{n,m} e^{mih}$. From (3),

this double series is absolutely convergent.

Interchanging the order of summation, we have

$$\sin(zf(h)) = \sum_{m=-\infty}^{\infty} \left(\sum_{n=1}^{\infty} \tfrac{1}{2}i((-iz)^n - (iz)^n)a_{n,m}/n! \right) e^{mih}$$

$$= \sum_{m=-\infty}^{\infty} c_m(z)\, e^{mih} \qquad (h \in H) \qquad (4)$$

where $c_m(z)$ is a constant depending only on z and m.

Now consider the function $g: R \to R$ defined by

$$g(t) = t \qquad (-\pi/2 \leq t \leq \pi/2)$$

$$g(t) = \pi - t \quad (\pi/2 \leq t \leq 3\pi/2)$$

$$g(t + 2\pi) = g(t) \qquad (t \in R).$$

Since g is continuous and periodic, it equals its Fourier series wherever the latter is convergent. As g is an odd function, the cosine terms have zero coefficients. Now

$$\int_{\pi/2}^{3\pi/2} (\pi - t)\sin nt\, dt = \int_{-\pi/2}^{\pi/2} -s \cos n\pi \sin ns\, ds$$

Hence $\int_{-\pi/2}^{3\pi/2} g(t) \sin nt\, dt = 0$ if n is even, and

$= 2\int_{-\pi/2}^{\pi/2} t \sin nt\, dt$ if n is odd. Assume now that

n is odd. Then $\int_{-\pi/2}^{\pi/2} t \sin nt \, dt =$

$$\left[\sin(nt)/n^2 - t\cos(nt)/n\right]_{-\pi/2}^{\pi/2} = 2\sin(n\pi/2)/n^2 .$$

Hence $g(t) = \frac{4}{\pi}\left(\sin(t) - \sin(3t)/3^2 + \dots\right)$.

Now putting $h = t1 = t$ in (1), where $t \in R$, we see that $f(t) = g(t)$. Given any odd integer p, it is easily verified from this that

$$\sin(pf(t)) = \sin(pt) \qquad (t \in R) \tag{5}$$

Then, from (4), $c_m(p) = (2\pi)^{-1}\int_0^{2\pi} \sin(pf(t))\, e^{-mit} dt$

$= (2\pi)^{-1}\int_0^{2\pi} \sin(pt)\, e^{-mit}\, dt$. Therefore, $-c_p(p) = c_{-p}(p) = \frac{1}{2}i$, and $c_m(p) = 0$ otherwise. Then, from (4) again,

$$\sin(pf(h)) = \sin(ph) \qquad (h \in H) \tag{6}$$

Now assume that $h \in H$, with $\rho(h) < c < \pi/2$. Let B be the maximal commutative subset of A containing 1 and h , so that B is a closed commutative subalgebra of A (Rickart [23] 1.6.14). Let ϕ be a non-zero multiplicative linear functional on B . Then, since $f(h) \in B$, and $e^{zh} \in B$ for $z \in C$, $\phi(f(h)) = \frac{4}{\pi}\left(\sin(\phi(h)) - \sin(3\phi(h))/3^2 + \dots\right) = \phi(h)$. Since $sp_A(f(h)) = sp_B(f(h))$, we have $\rho(f(h)) = \rho(h) < c$. By lemma 7, we can therefore find $K > 0$ such that $\|e^{zh}\| < Ke^{c|z|}$ and $\|e^{zf(h)}\| < Ke^{c|z|}$ $(z \in C)$.
Define a function $k\colon C \to A$ by

$$k(z) = \sin((2z+1)f(h)) - \sin((2z+1)h) \qquad (z \in C)$$

Now $\|\sin(2z+1)h\| \leq \frac{1}{2}\|e^{ih}\ e^{2izh}\| + \frac{1}{2}\|e^{-ih}\ e^{-2izh}\| \leq$ $Ke^{2c|z|}$ $(z \in C)$. Similarly, $\|\sin((2z+1)f(h))\| \leq$ $Ke^{2c|z|}$ $(z \in C)$. Hence $\|k(z)\| \leq 2Ke^{2c|z|}$ $(z \in C)$. Also, $k(z)$ is an entire function, $2c < \pi$, and, from (6) $k(z) = 0$ $(z = 0,1,2,\dots)$. Therefore, by lemma 9, $k(z) = 0$ $(z \in C)$. Hence $\sin(zh) = \sin(zf(h))$ $(z \in C)$. Comparing terms in z in these entire functions, we must have $h = f(h)$. Hence

$$h = \frac{4}{\pi}\left(\sin(h) - \sin(3h)/3^2 + \dots \right) \quad (h \in H,\ \rho(h) < \tfrac{1}{2}\pi)$$

Now assume that $h \in H$ and $\rho(h) = \frac{1}{2}\pi$. Let $0 < r < 1$. Then $rh \in H$, $\rho(rh) < \frac{1}{2}\pi$, and we have

$$rh = \frac{4}{\pi}\left(\sin(rh) - \sin(3rh)/3^2 + \dots \right)$$

For $z \in C$, $\sin(zrh) \to \sin(zh)$ as $r \to 1$. Using this and the absolute convergence of the series for rh, we have, on letting $r \to 1$,

$$h = \frac{4}{\pi}\left(\sin(h) - \sin(3h)/3^2 + \dots \right)$$

From (2), we have $h^2 = f(h)^2 = \sum_{-\infty}^{\infty} a_{2,m}\ e^{mih}$. The coefficients $a_{2,m}$ may be calculated from

$$2\pi\, a_{2,m} = \int_0^{2\pi} f(t)^2\ e^{-mit}\ dt .$$

When this is done, we have

$$h^2 = \pi^2/12 - \cos(2h) + \cos(4h)/2^2 - \dots \quad (\rho(h) = \tfrac{1}{2}\pi)$$

We now use this theorem to prove part of Sinclair's result.

COROLLARY 4.11. Let A be a complex Banach algebra with identity of norm one. Let $h \in A$ be Hermitian.

Then $\|h\| = \rho(h)$.

Proof. Let $r > 0$ be such that $\rho(rh) \leq \frac{1}{2}\pi$. Then, since rh is Hermitian, theorem 10 shows that

$$rh = \frac{4}{\pi}\left(\sin(rh) - \sin(3rh)/3^2 + \dots \right)$$

Since $\|\sin(th)\| \leq 1$ for $t \in R$, we have

$$\|rh\| \leq \frac{4}{\pi}\left(1 + 1/3^2 + 1/5^2 + \dots\right) = \tfrac{1}{2}\pi.$$

In the case $\rho(h) = 0$, $r > 0$ is arbitrary, and so $h = 0$. If $\rho(h) > 0$, taking $r = \frac{1}{2}\pi/\rho(h)$ gives $\|h\| \leq \rho(h)$. Since always $\|h\| \geq \rho(h)$, the result is proved.

With A as in theorem 10, and $h \in H$ with $\rho(h) = 1$, we have by theorem 10,

$$h = -4i\pi^{-2}\left(\dots + e^{-3k}/3^2 - e^{-k} + e^{k} - e^{3k}/3^2 + \dots \right)$$

where $k = \frac{1}{2}i\pi h$. This is an expression of h as a convex sum of elements of the form e^{irh} with $r \in R$. It is natural to ask whether this can be done for $h + z$, where $\rho(h+z) = 1$. We now describe a situation in which this is possible.

Let $t \in R$. Then, since $\rho(h) = 1$ and $sp(h) \subset R$, we have $\rho(\cos t + i \sin t.h) = 1$. Let B be the maximal commutative subset of A containing 1 and h, so that B is a closed subalgebra. Suppose that B is semi-simple. Let f be a non-zero multiplicative linear functional on B. Then, since $-1 \leq f(h) \leq 1$, we have for $0 < t < \pi$, using Fourier series theory as in the proof of theorem 10,

$$f(e^{-ith}(\cos t + i \sin t.h)) = e^{-itf(h)}(\cos t + i\sin t.f(h))$$

$$= \sum_{-\infty}^{\infty} \frac{(-1)^n \sin^2 t}{(n\pi + t)^2} e^{ni\pi} f(h)$$

$$= f\left(\sum_{-\infty}^{\infty} \frac{(-1)^n \sin^2 t}{(n\pi + t)^2} e^{ni\pi h} \right)$$

Since this is true for any element of the carrier space of B, we have

$$e^{-ith}(\cos t + i \sin t.h) = \sum_{-\infty}^{\infty} \frac{(-1)^n \sin^2 t}{(n\pi + t)^2} e^{ni\pi h} \quad , \text{ or}$$

$$\cos t + i \sin t.h = \sum_{-\infty}^{\infty} \frac{(-1)^n \sin^2 t}{(n\pi + t)^2} e^{(n\pi + t)ih}$$

This is an expression for $\cos t + i \sin t.h$ as a convex sum of elements e^{irh} with $r \in R$, as is seen by putting $h = 1$.

Suppose that X is a finite dimensional complex normed space, and that T is a Hermitian operator on X. The eigenvalues of T, being on the boundary of $V(T)$, must have ascent one, by theorem 2.5. Hence, relative to a suitable basis, T may be expressed as a real diagonal matrix. A consequence of this is that the algebra generated by I and T is semi-simple. Hence, in this case, assuming $\sigma(T) = 1$ and $t \in R$, we can express $\cos t + i \sin t.T$ as a convex sum of operators e^{irT}, with $r \in R$.

We now give another consequence of theorem 10.

THEOREM 4.12. Let A be a complex Banach algebra with identity of norm one. Then, for $h \in A$ and h Hermitian,

$$\|h^2 - t\| = \rho(h^2 - t) \qquad (0 \leq t \leq \ (h^2)/3)$$

Proof. Assume first that $\rho(h) = \frac{1}{2}\pi$. Then by theorem 10,

$$h^2 = \pi^2/12 - \cos(2h) + \cos(4h)/2^2 - \ldots \qquad (1)$$

Suppose that $0 \leq t \leq \pi^2/12 = \rho(h^2)/3$. Then from (1)

$$\|h^2 - t\| \leq \pi^2/12 - t + \pi^2/6 = \pi^2/4 - t \quad .$$

Since $\rho(h) = \frac{1}{2}\pi$, either $\frac{1}{2}\pi \in sp(h)$ or $-\frac{1}{2}\pi \in sp(h)$. In either case, $\frac{1}{4}\pi^2 - t \in sp(h^2 - t)$. Hence $\rho(h^2 - t) \geq \frac{1}{4}\pi^2 - t \geq \|h^2 - t\| \geq \rho(h^2 - t)$. Therefore $\|h^2 - t\| = \rho(h^2 - t)$. The result for general h follows by considering $\frac{1}{2}\pi h/\rho(h)$.

It is clear that $\|h^2 + t\| = \rho(h^2 + t)$ for h Hermitian and $t > 0$. An argument similar to that of theorem 12 will show that $\|h^3 - th\| = \rho(h^3 - th)$ for $0 \leq t \leq 3(1 - 8/\pi^2)\rho(h)^2$, and other such results.

The next example shows that the equality of theorem 12 need not hold when $t > \rho(h^2)/3$.

EXAMPLE 4.13. Let X be C^5 with the norm

$$p(x_1, x_2, x_3, x_4, x_5) = \sup\left\{ |z^{-2}x_1 + z^{-1}x_2 + x_3 + zx_4 + z^2x_5| : |z| = 1 \right\}.$$

Define $T \in B(X)$ by $T(x_1, x_2, x_3, x_4, x_5) =$

$(-2x_1, -x_2, 0, x_4, 2x_5)$. Let $R = T^2 - 2I$. Then T is Hermitian, and $\|R\| > \rho(R)$.

Proof. For any positive integer n we have

$$T^n(x_1, x_2, x_3, x_4, x_5) = ((-2)^n x_1, (-1)^n x_2, 0, x_4, 2^n x_5) .$$

Hence for $z \in C$,

$$e^{zT}(x_1, \ldots, x_5) = (e^{-2z}x_1, e^{-z}x_2, x_3, e^z x_4, e^{2z}x_5) .$$

Therefore for $a \in R$,

$$\begin{aligned} p(e^{iaT}(x_1, \ldots, x_5)) &= p(e^{-2ia}x_1, \ldots, e^{2ia}x_5) \\ &= p(x_1, \ldots, x_5) . \end{aligned}$$

From this, $p(e^{iaT}) = 1$ $(a \in R)$, and so T is Hermitian. Now let x be the point $(1, -i, 1, i, 1)$. Suppose that $z = e^{it}$, with $t \in R$. Then

$$z^{-2} - iz^{-1} + 1 + iz + z^2 = 2\cos 2t - 2\sin t + 1$$
$$= 2(1 - 2\sin^2 t) - 2\sin t + 1 = 13/4 - 4(\sin t + \tfrac{1}{4})^2 .$$

From this, it is easy to see that $p(x) = 13/4$. Now $Rx = (2, i, -2, -i, 2)$, so that

$$p(Rx) \geq |2(-i)^{-2} + i(-i)^{-1} + (-2) + (-i)(-i) + 2(-i)^2|$$
$= 8$. Hence $p(R) \geq 32/13 > 2 = \rho(R)$.

If in example 13 T^2 were Hermitian, then R would be Hermitian, and we would have $\|R\| = \rho(R)$ by theorem 8. Hence we have an example of a Hermitian operator whose square is not Hermitian. In example 14 we give a more straightforward example of this. In [18] Lumer uses the theory of spectral operators on infinite dimensional spaces to prove the existence of Hermitian operators whose squares are not Hermitian.

EXAMPLE 4.14. Let X be $\mathbb{C}^3$ with the norm $p(x,y,z) = \sup\{|w^{-1}x + y + wz| : |w| = 1\}$. Define $T \in B(X)$ by $T(x,y,z) = (-x,0,z)$. Then T is Hermitian, but T^2 is not Hermitian.

Proof. For $a \in R$, $p(e^{iaT}(x,y,z)) = p(e^{-ia}x,y,e^{ia}z) = p(x,y,z)$, so that $p(e^{iaT}) = 1$. Hence T is Hermitian.

Now $T^2(x,y,z) = (x,0,z)$, and $e^{isT^2}(x,y,z) = (ix,y,iz)$ where $s = \frac{1}{2}\pi$. Therefore $p(e^{isT^2}(1,i,1)) = p(i,i,i) = 3$, while $p(1,i,1) = 5^{\frac{1}{2}}$. Hence $p(e^{isT^2}) > 1$, and T^2 is not Hermitian.

Remark. In this example we also have $I+T$ Hermitian, while $(I+T)^n$ is not Hermitian for $n > 1$. To see this, note that $T^3 = T$, and use the binomial theorem.

Although T Hermitian does not imply that T^2 is Hermitian, we can show that $V(T^2)$ lies to the right of the imaginary axis.

THEOREM 4.15. Suppose that T is Hermitian. Then $\operatorname{Re} z \geq 0$ for $z \in V(T^2)$.

Proof. Let $x \in S$ and $f \in D(x)$. Then, for $t \in R$, $\|(I + itT)x\| \geq |f((I + itT)x)| = |1 + itf(Tx)| \geq 1$. Therefore $\|(I + itT)x\| \geq \|x\|$ $(x \in X)$, and so $\|(I + t^2T^2)x\| = \|(I - itT)(I + itT)x\| \geq \|x\|$ $(x \in X)$. Hence, for $t > 0$, $\|(I + tT^2)^{-1}\| \leq 1$, and

$$\|\exp(-tT^2)\| = \|\lim_{n\to\infty} (I + tT^2/n)^{-n}\| \leq 1.$$ Finally,

by theorem 1.9, $\sup \operatorname{Re} V(-T^2) =$

$$\sup \left\{ t^{-1} \log\|\exp(-tT^2)\| : t > 0 \right\} \leq 0.$$

Following Lumer [18], an operator T of the form $T = R + iJ$, with R and J commuting Hermitian operators is said to be <u>normal</u>. The following result on normal operators was proved by T.W. Palmer [21].

THEOREM 4.16. For a normal operator T, $v(T) = \rho(T)$.

This result, together with the fact that $\|T + z\| = \rho(T + z)$ for T Hermitian and $z \in C$, suggests that the equality $\|T\| = \rho(T)$ might hold for normal T. Example 17, however, gives a normal operator T with $\|T\| = 2^{\frac{1}{2}}\rho(T)$.

EXAMPLE 4.17. Let A be the algebra of elements
$q + ru + sv + tuv \qquad (q,r,s,t \in C)$,
where $u^2 = v^2 = 1$, and $uv = vu$. For $a \in A$, define

$$p(a) = \inf \left\{ \sum_{k=1}^{n} |c_k| : \sum_{k=1}^{n} c_k(\cos s_k + i\sin s_k u)(\cos t_k + i\sin t_k v) = a,\ c_k \in C,\ s_k,\ t_k \in R,\ n \in Z \right\}$$

Then p is an algebra-norm on A, u and v are Hermitian, and $p(u+iv) = 2^{\frac{1}{2}}\rho(u+iv)$.

<u>Proof</u>. Suppose $a,b \in A$. Given any $\epsilon > 0$, we can find a positive integer m, and $c_k \in C$, q_k, $r_k \in R$ $(k = 1,\dots,m)$ such that

$$\sum_{j=1}^{m} c_j(\cos q_j + i\sin q_j u)(\cos r_j + i\sin r_j v) = a \qquad (1)$$

and

$$\sum_{j=1}^{m} |c_j| < p(a) + \epsilon . \qquad (2)$$

Similarly, $\sum_{k=1}^{n} d_k(\cos s_k + i\sin s_k u)(\cos t_k + i\sin t_k v) =$ b, and $\sum_{k=1}^{n} |d_k| < p(b) + \epsilon$. Using the fact that $(\cos s + i\sin s.u)(\cos t + i\sin t.u) = \cos(s+t)$ $+ i\sin(s+t)u$, and the similar identity for v, we have

$$\sum_{j=1}^{m} \sum_{k=1}^{n} c_j d_k(\cos(q_j + s_k) + i\sin(q_j + s_k)u).$$
$$(\cos(r_j + t_k) + i\sin(r_j + t_k)v) = ab,$$

so that $p(ab) \leq \sum_{j=1}^{m} \sum_{k=1}^{n} |c_j d_k| < (p(a) + \epsilon)(p(b) + \epsilon)$. Since ϵ is arbitrary, $p(ab) \leq p(a)p(b)$. Also, since

$$a + b = \sum_{j=1}^{m} c_j(\cos q_j + i\sin q_j u)(\cos r_j + i\sin r_j v) +$$
$$\sum_{k=1}^{n} d_k(\cos s_k + i\sin s_k u)(\cos t_k + i\sin t_k v),$$

we have $p(a+b) \leq \sum_{j=1}^{m} |c_j| + \sum_{k=1}^{n} |d_k| \leq p(a)+p(b)+2\epsilon$. Since ϵ is arbitrary, $p(a+b) \leq p(a) + p(b)$. Now suppose that $p(a) = 0$, where $a = q + ru + sv + tuv$. Then, from (1) and (2), $|q| = |\sum_{j=1}^{m} c_j \cos q_j \cos r_j|$ $\leq \sum_{j=1}^{m} |c_j| < \epsilon$. Since ϵ is arbitrary, $q = 0$. Similarly, $r = s = t = 0$, so that $a = 0$. Finally, we show that $p(1) = 1$. For $1 = (\cos 0 + i\sin 0.u)$. $(\cos 0 + i\sin 0.v)$, so that $p(1) \leq 1$. Also, putting

$a = 1$ in (1), $\sum_{j=1}^{m} |c_j| \geq \sum_{j=1}^{m} |c_j \cos q_j \cos r_j| \geq 1$,

so that $p(1) \geq 1$.

We have now established that p is an algebra-norm on A, with $p(1) = 1$. For $t \in R$, it is clear that $e^{itu} = \cos t + i\sin t.u = (\cos t + i\sin t.u).(\cos 0 + i\sin 0,v)$, so that $p(e^{itu}) \leq 1$. Hence u, and similarly v, is Hermitian. Also, $p(u) = \rho(u) = 1 = p(v)$.

Suppose that $u + iv =$

$$\sum_{k=1}^{n} c_k(\cos s_k + i\sin s_k u)(\cos t_k + i\sin t_k v)$$

with $c_k \in C$, s_k, $t_k \in R$ $(k = 1,\ldots,n)$. Then

$$\sum_{k=1}^{n} c_k\cos s_k \sin t_k = 1 \text{ and } \sum_{k=1}^{n} c_k \sin s_k \cos t_k = -i\ .$$

Define $a_k = (1+i)c_k$ $(k = 1,\ldots,n)$. Then

$$\sum_{k=1}^{n} a_k\sin(s_k + t_k) = 2 \text{ and } \sum_{k=1}^{n} a_k\sin(t_k - s_k) = 2i\ .$$

Hence, $\sum_{k=1}^{n} |\mathrm{Re}\ a_k| \geq \sum_{k=1}^{n} |\mathrm{Re}\ a_k\sin(s_k + t_k)| \geq$

$\sum_{k=1}^{n} \mathrm{Re}\ a_k\sin(s_k + t_k) = 2$. Similarly, $\sum_{k=1}^{n} |\mathrm{Im}\ a_k| \geq 2$.

Therefore

$$2\sum_{k=1}^{n} |c_k| = 2^{\frac{1}{2}} \sum_{k=1}^{n} |a_k| \geq \sum_{k=1}^{n} |\mathrm{Re}\ a_k| + |\mathrm{Im}\ a_k| \geq 4.$$

Hence $p(u+iv) \geq 2$. Also, $p(u+iv) \leq p(u) + p(v) = 2$, so that $p(u+iv) = 2$. Finally, $(u+iv)^2 = 2iuv$, and $(uv)^2 = uv$, so that $\rho(u+iv) = 2^{\frac{1}{2}}$. Therefore, $p(u+iv) = 2^{\frac{1}{2}}\rho(u+iv)$.

J. Duncan has pointed out that, for normal T, we always have $\|T\| \leq 2\rho(T)$. For if $T = R + iJ$, with R, J commuting Hermitians, then $\|T\| \leq \|R\| + \|J\| = \rho(R) + \rho(J) \leq 2\rho(T)$. The best constant M in $\|T\| \leq M\rho(T)$ (T normal) is not known, but must lie between $2^{\frac{1}{2}}$ and 2.

The theory of Hermitian elements of a normed algebra may be applied to part of a problem considered in chapter 3. The following result is due to Lumer.

Let A be a real Banach algebra, with identity of norm one. Let $a \in A$ have $v(a) = 0$. Let B be the complexification of A, and let b be the element of B corresponding to $a \in A$. Then, as in the proof of theorem 3.12, $\sup\{|\operatorname{Re} z| : z \in V(b)\} = v(a) = 0$. Hence $V(b) \subset iR$, so that ib is Hermitian. Also, $sp(b) \subset iR$, and therefore $sp(b^2) \subset R$. Therefore

$$\begin{aligned} v(a^2) &= \sup\{|\operatorname{Re} z| : z \in V(b^2)\} \\ &\geq \sup\{|\operatorname{Re} z| : z \in sp(b^2)\} \\ &= \rho(b^2) = \rho(b)^2 = \|b\|^2 = \|a\|^2 . \end{aligned}$$

Hence $\|a\| \leq v(a^2)^{\frac{1}{2}}$. The result of Bonsall and Duncan that $v(a) = v(a^2) = 0$ implies $a = 0$, follows immediately from this.

Still assuming $v(a) = 0$, clearly $sp(b^n) \subset R$ for any even integer n. Arguing as above, we then have $\|a\| \leq v(a^n)^r$, where $r = 1/n$. It is not known whether an inequality of the form $\|a\| \leq pv(a) + qv(a^n)^r$ with p and q constants, exists for n an even integer greater than 2.

1. F.L. Bauer, 'On the field of values subordinate to a norm', Numer. Math. 4(1962) 103 - 111.

2. C.A. Berger and J.G. Stampfli, 'Mapping theorems for the numerical range', Amer. J. Math. 89(1967) 1047 - 1055.

3. R.P Boas, 'Entire Functions', Academic Press, New York, 1954.

4. H.F. Bohnenblust and S. Karlin, 'Geometrical properties of the unit sphere of Banach algebras', Ann. of Math. 62(1955) 217 - 229.

5. F.F. Bonsall, 'The numerical range of an element of a normed algebra', Glasgow Math. J. 10(1969) 68 - 72.

6. F.F. Bonsall, B.E. Cain, and H. Schneider, 'The numerical range of a continuous mapping of a normed space', Aequationes Math. 2(1968) 86 - 93.

7. F.F. Bonsall and J. Duncan, 'Dually irreducible representations of Banach algebras', Q.J. Math. (Oxford) 19(1968) 97 - 111.

8. F.F. Bonsall and J. Duncan, 'Numerical Range', London Math. Soc. Lecture Note Series.

9. S.R. Caradus, 'On meromorphic operators, I', Canad. J. Math. 19(1967) 723 - 736.

10. J.A. Clarkson, 'Uniformly convex spaces', Trans.

Amer. Math. Soc. 40(1936) 396 - 414.

11. M.J. Crabb, 'Numerical range estimates for the norms of iterated operators' (to appear in Glasgow Math. J.).

12. M.J. Crabb, 'Some results on the numerical range of an operator', (to appear in J. London Math. Soc.).

13. J. Duncan, C.M. McGregor, J.D. Pryce, and A.J. White, 'The numerical index of a normed space', (to appear in J. London Math. Soc.).

14. B.W. Glickfeld, 'On an inequality of Banach algebra geometry and semi-inner-product space theory, (to appear in Ill. J. Math.).

15. F. Hausdorff, 'Der Wertvorrat einer Bilinearform', Math. Zeit. 3(1919) 314 - 316.

16. S. Kakutani, 'A generalisation of Brouwer's fixed point theorem', Duke Math. J. 8(1941) 457 - 459.

17. G. Lumer, 'Semi-inner-product spaces', Trans. Amer. Math. Soc. 100(1961) 29 - 43.

18. G. Lumer, 'Spectral operators, Hermitian operators and bounded groups', Acta Sci. Math. (Szeged) 25(1964) 75 - 85.

19. G. Lumer and R.S. Phillips, 'Dissipative operators in a Banach space', Pac. J. Math.11(1961)679-698.

20. M. Nirschl and H. Schneider, 'The Bauer fields of values of a matrix', Numer. Math. 6(1964 355 - 365.

21. T.W. Palmer, 'Unbounded normal operators on Banach spaces', Trans. Amer. Math. Soc. 133(1968) 385 - 414.

22. C. Pearcy, 'An elementary proof of the power inequality for the numerical radius', Mich. Math. J. 13(1966) 289 - 291.

23. C.E. Rickart, 'General theory of Banach algebras', Van Nostrand, (1960).

24. A.M. Sinclair, 'The norm of a Hermitian element in a Banach algebra', (to appear).

25. A.E. Taylor, 'Mittag Leffler expansions and spectral theory,' Pacific J. Math. 10,3(1960) 1049 - 1066.

26. E.C. Titchmarsh, 'The theory of functions', Oxford University Press, 1944.

27. O. Toeplitz, 'Das algebraische Analogon zu einem Satze von Fejer', Math. Zeit. 2(1918) 187 - 197.

28. I. Vidav, 'Eire metrische Kennzeichnung der selbstadjungierten Operatoren', Math Zeit. 66(1956) 121 - 128.

29. J. P. Williams, 'Spectra of products and numerical ranges', J. Math. Anal. and Appl. 17(1967) 214 - 220.

30. J.P. Williams and T. Crimmins, 'On the numerical radius of a linear operator', Amer. Math. Monthly 74(1967) 832 - 833.

31. Chr. Zenger, 'On convexity properties of the Bauer field of values of a matrix', Num. Math. 12 (1968) 96 - 105.